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Unavoidable doubly connected large graphs☆

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Abstract

A connected graph is *doubly connected* if its complement is also connected. The following Ramsey-type theorem is proved in this paper. There exists a function $h(n)$, defined on the set of integers exceeding three, such that every doubly connected graph on at least $h(n)$ vertices must contain, as an induced subgraph, a doubly connected graph, which is either one of the following graphs or the complement of one of the following graphs:

(1) P_n, a path on n vertices;
(2) $K_{1,n}^s$, the graph obtained from $K_{1,n}$ by subdividing an edge once;
(3) $K_{2,n}\backslash e$, the graph obtained from $K_{2,n}$ by deleting an edge;
(4) $K_{2,n}^+$, the graph obtained from $K_{2,n}$ by adding an edge between the two degree-n vertices x_1 and x_2, and a pendent edge at each x_i.

Two applications of this result are also discussed in the paper.

Keywords: Unavoidable graph; Cograph; Doubly connected graph; Induced subgraph; Well quasi order

1. Introduction

All graphs considered in this paper are finite and simple. We follow [12] for our terminology. In particular, the *complement* of a graph G will be denoted by $\bar{G}$. We begin with a classical result of Ramsey [9].

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Ramsey's Theorem. *There exists a function $r(n)$, defined on the set of positive integers, such that every graph on at least $r(n)$ vertices must contain either K_n or $\bar{K}_n$ as an induced subgraph.*

In graph theory, there are many results that are similar to Ramsey's Theorem and they are known as Ramsey-type theorems. These results claim that if a graph G with certain property is large enough, then G must contain a relatively large graph H, such that H still has the same property but H is better structured than G. For instance, in Ramsey's Theorem, it is clear that both K_n and its complement $\bar{K}_n$ are better structured than the general graph G.

In Ramsey's Theorem, the graphs that are in consideration are not required to have any special properties other than being big. The next is a Ramsey-type result where the property we are interested in is being connected. As usual, a path on n vertices is denoted by P_n.

1.1. *There exists a function $r_c(n)$, defined on the set of positive integers, such that every connected graph on at least $r_c(n)$ vertices must contain a connected graph K_n, P_n, or $K_{1,n}$, as an induced subgraph.*

This result is an easy consequence of Ramsey's Theorem. For the sake of completeness, a proof is given in the next section. A different way to formulate (1.1) is to claim that, for every n, at least one of K_n, P_n, and $K_{1,n}$ is unavoidable, as an induced subgraph, in every sufficiently large connected graph. For 2-, 3-, and 4-connectivity, there are results [8] analogous to (1.1). There are also similar results on matroids (see [4,5]), which we do not discuss here.

A graph is *doubly connected* if its complement is also connected. For example, the path P_n is doubly connected, when $n \geqslant 4$. On the other hand, the complete bipartite graph $K_{m,n}$ is connected but not doubly connected, as its complement has two connected components, K_m and K_n. The main problem we are going to consider in this paper is: what are the unavoidable doubly connected large induced subgraphs in a sufficiently large doubly connected graph?

Let n be a positive integer. Let $K^s_{1,n}$ be the graph obtained from $K_{1,n}$ by subdividing an edge once; let $K_{2,n}\backslash e$ be the graph obtained from $K_{2,n}$ by deleting an edge; furthermore, let $K^+_{2,n}$ be the graph obtained from $K_{2,n}$ by adding an edge between the two degree-n vertices x_1 and x_2, and, for $i = 1,2$, a pendent edge at x_i. These graphs, together with P_n, are illustrated in Fig. 1 below, for $n = 5$.

For each positive integer n, let

$$\mathcal{U}_n = \{P_n, \overline{P_n}, K^s_{1,n}, \overline{K^s_{1,n}}, K_{2,n}\backslash e, \overline{K_{2,n}\backslash e}, K^+_{2,n}, \overline{K^+_{2,n}}\}.$$

Then it is straightforward to verify that, when $n \geqslant 4$, all graphs in $\mathcal{U}_n$ are doubly connected. The following, our first main result in this paper, says that graphs in $\mathcal{U}_n$ are unavoidable, as induced subgraphs, in every sufficiently large doubly connected graph.

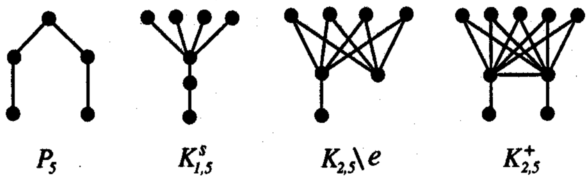

Fig. 1. Unavoidable doubly connected graphs.

1.2 Theorem. *There exists a function $h(n)$, defined on the set of positive integers, such that every doubly connected graph on at least $h(n)$ vertices must contain a graph in $\mathscr{U}_n$ as an induced subgraph.*

From an application point of view, (1.2) can be formulated in a different way, which is explained below. We begin with some definitions. The subgraph of a graph G induced by a set X of vertices is denoted by $G[X]$. The *disjoint union* of two graphs G_1 and G_2 is a graph G, for which $V(G)$ can be partitioned into X_1 and X_2, such that G has no edges between X_1 and X_2, and, for $i = 1, 2$, the induced subgraph $G[X_i]$ is isomorphic to G_i.

Let $\mathscr{G}$ be a class of graphs. We define $\mathscr{G}^*$ to be the class of graphs that can be constructed, starting from graphs in $\mathscr{G}$, by repeatedly taking disjoint unions and taking complements. Let us call a class of graphs *closed* if the complement of any member remains a member, and the disjoint union of any two members also remains a member. Then the following is an equivalent definition of $\mathscr{G}^*$.

1.3. *$\mathscr{G}^*$ is the smallest closed class that contains $\mathscr{G}$.*

The proof of this proposition is easy, and it is given in the next section for completeness. For each positive integer n, let $\mathscr{G}_n$ be the class of graphs that do not contain any graph in $\mathscr{U}_n$ as an induced subgraph. Then the following is a reformulation of (1.2).

1.4. *Each $\mathscr{G}_n$ can be expressed as $\mathscr{G}^*$, for some finite $\mathscr{G}$.*

The equivalence of (1.4) and (1.2) will be proved in the next section. In the following, we discuss two nice applications of (1.4). A graph property **P** is *hereditary* if the induced subgraphs of a graph that has property **P** must also have property **P**. For instance, being a complete graph is hereditary. It is well-known, and it is also very easy to show, that, for every hereditary graph property **P**, there exists a unique minimal (under inclusion) set $\mathscr{C}$ of graphs, such that a graph has property **P** if and only if the graph does not contain any graph in $\mathscr{C}$ as an induced subgraph. There are many results in graph theory that determine $\mathscr{C}$ for various **P**. These results are interesting theoretically, but they do not always have algorithmic implications since $\mathscr{C}$ could be infinite. In the next theorem, which is our second main result in this paper, we describe a class of hereditary graph properties, for which the corresponding $\mathscr{C}$ is guaranteed to be finite.

1.5 Theorem. *Let* **P** *be a hereditary graph property and let n be a positive integer. Suppose no graph in* $\mathscr{U}_n$ *has property* **P**. *Then there exist finitely many graphs* $G_1, G_2, \ldots, G_k$ *such that a graph has property* **P** *if and only if the graph does not contain any* G_i *as an induced subgraph.*

Clearly, if **P** is a graph property as described in (1.5), then the problem of deciding if a graph G has property **P** is equivalent to the problem of testing if G contains any G_i as an induced subgraph. Since, for any fixed graph H, the problem of testing if H is an induced subgraph can be solved in polynomial time, we conclude from (1.5) the following.

1.6. *Suppose* **P** *is a graph property as described in* (1.5). *Then the problem of deciding if a graph has property* **P** *can be solved in polynomial time.*

There are two remarks that we would like to make about (1.6). First, (1.6) is a very general result since **P** is a general graph property, which is only required to satisfy certain very weak conditions. On the other hand, (1.6) only tells us the existence of a polynomial time algorithm, it does not tell us how to construct such an algorithm. In fact, our proof of (1.5) does not give us this information either because it is non-constructive.

Next, we consider another application of (1.4), from which we will have our third main result in this paper. This is about the structure of $\mathscr{G}^*$ when $\mathscr{G}$ is finite. We begin with an explanation on why we are interested in this problem. Suppose $\mathscr{G}=\{K_1\}$. Then the class $\mathscr{G}^*$ is known as the class of *cographs* [2], which was first introduced in [7] and was also characterized in the same paper as follows.

1.7. *A graph is a cograph if and only if it does not contain* P_4 *as an induced subgraph*

Cographs have been rediscovered several times by different researchers, under various names, including *dacey graphs* [10], D^*-*graphs* [6], and 2-*parity graphs* [1]. Such a broad interest in these graphs naturally suggests the following question.

1.8 Question. *When can a class of graphs be expressed as* $\mathscr{G}^*$ *for some finite* $\mathscr{G}$?

It turns out that reformulation (1.4) of our first main result (1.2) provides an answer to this question. We will present three answers with the first being a partial answer. This answer is much more clean than the next two and it is more or less equivalent to (1.4).

1.9. *A class* $\mathscr{H}$ *of graphs is contained in* $\mathscr{G}^*$ *for some finite* $\mathscr{G}$ *if and only if it is contained in some* $\mathscr{G}_n$.

The next is a complete answer to Question (1.8).

1.10. *Let $\mathscr{H}$ be a class of graphs. Then $\mathscr{H} = \mathscr{G}^*$ for some finite $\mathscr{G}$ if and only if.*

(1) $\mathscr{H}$ is closed;
(2) $\mathscr{H} \subseteq \mathscr{G}_n$, *for some n; and*
(3) *if $\mathscr{S} \subseteq \mathscr{H}$ contains infinitely many pairwise non-isomorphic disconnected graphs, then some graph in $\mathscr{S}$ is the disjoint union of two other graphs in $\mathscr{H}$.*

By applying (1.5), the last result can be refined. An infinite sequence $G_1, G_2, \ldots$ of graphs is *monotone* if each G_i is a proper induced subgraph of G_{i+1}. Now our third main result in this paper can be stated as follows.

1.11 Theorem. *Let $\mathscr{H}$ be a class of graphs. Then $\mathscr{H} = \mathscr{G}^*$ for some finite $\mathscr{G}$ if and only if:*

(1) *$\mathscr{H}$ is closed;*
(2) *$\mathscr{H} \subseteq \mathscr{G}_n$, for some n; and*
(3) *if $G_1, G_2, \ldots$ is a monotone sequence of disconnected graphs in $\mathscr{H}$, then some G_i is the disjoint union of two other graphs in $\mathscr{H}$.*

Finally, we point out a connection between (1.7) and our first main result (1.2). Notice that the main part of (1.7) is the "if" direction, which claims that, *if a graph G on two or more vertices does not contain P_4 as an induced subgraph, then either G or $\bar{G}$ is disconnected.* Meanwhile, (1.2) can be formulated similarly as: *if a graph G on $h(n)$ or more vertices does not contain any graph in $\mathscr{U}_n$ as an induced subgraph, where n is a positive integer, then either G or $\bar{G}$ is disconnected.* From this point of view, we can say that (1.2) is a generalization of (1.7).

We close this section by outlining the rest of the paper. In Section 2, we prove our first main result, (1.2), as well as its equivalent formulation (1.4). Proofs of (1.1) and (1.3) are also given in this section. Then, in Section 3, we prove (1.5), our second main result, by showing that $\mathscr{G}_n$ is well quasi ordered under the induced subgraph relation. Finally, in Section 4, we prove (1.11), our third main result, and two weaker versions, (1.9) and (1.10), of this result.

2. Proving the first main result

Recall that $r(n)$ is the function defined in Ramsey's Theorem. In this section, for each positive integer n, we also need the following function:

$$p_n(x) = 1 + x + x^2 + \cdots + x^{n-1}.$$

Let P be an induced path of a graph G. Let $B = G - V(P)$ and let the ends of P be u and v. Notice that u and v are identical if $P = K_1$. In addition, B is an empty graph if $V(P) = V(G)$. Suppose u is adjacent to all vertices of B and no other vertices of P are adjacent to any vertices of B. Then we call G a *tadpole graph*, with *tail* P and *body B*. We also call v the *tip* of its tail. The next is a simple observation, which will be used more than once in our proofs.

2.1. *Let k and n be a positive integers and let v be a vertex of a connected graph G.* If $|V(G)| > p_n(k)$, *then G contains an induced tadpole graph with v as the tip of its tail and such that either its tail has more than n vertices or its body has more than k vertices.*

Proof. Let T be a breadth-first search tree rooted at v. That is, T is a spanning tree of G such that, for each vertex u of G, the unique uv-path in T is a shortest uv-path in G. It is standard to call a vertex w a *child* of a vertex u if $uw \in E(T)$ and u is contained in the unique wv-path in T. For each vertex u of G, let X_u be the set of all children of u and let Y_u be the set of vertices in the unique uv-path in T. Then it is easy to see that $G_u = G[X_u \cup Y_u]$ is a tadpole graph with body $G(X_u)$, tail $G(Y_u)$, and tip of its tail v.

If there is a vertex u with more than k children, then the tadpole graph G_u has the required properties as X_u has more than k vertices. Therefore, we may assume that each vertex can have at most k children. For each integer $t \geqslant 0$, let N_t be the set of vertices that are distance t away from v. It is clear that $|N_0| = 1$, and $|N_t| \leqslant k|N_{t-1}|$, for all positive integers t. Since $|V(G)| > p_n(k)$ and G is connected, N_n must contain at least one vertex, say u. Now it is clear that the tadpole graph G_u has the required properties as Y_u has more than n vertices. □

The following is an immediate consequence of (2.1).

2.2. *Let k and n be a positive integers and let G be a connected graph. If $|V(G)| > p_n(k)$, then G contains either an induced P_{n+1} or a vertex of degree greater than k.*

Proof of (1.1). Let $r_c(1) = 1$ and, for $n \geqslant 2$, $r_c(n) = 1 + p_{n-1}(k)$, where $k = r(n) - 1$. Let G be a connected graph with at least $r_c(n)$ vertices. We need to show that G has K_n, P_n, or $K_{1,n}$ as an induced subgraph. First, notice that, if $n = 1$, then $|V(G)| \geqslant 1$ and thus G contains K_1 as an induced subgraph. Therefore, we may assume in the following that $n \geqslant 2$.

If the maximum degree of G is at most k, by (2.2), G must contain P_n as an induced subgraph. In this case, Proof of (1.1) is proved. Next, we consider the case when some vertex of G, say x, is adjacent with a set X of $k + 1 = r(n)$ vertices. By applying Ramsey's Theorem to $G[X]$, we conclude that X has a subset X' such that $G[X']$ is either K_n or $\bar{K}_n$. It follows that either $G[X'] = K_n$ or $G[X' \cup \{x\}] = K_{1,n}$, both satisfy the conclusion of (1.1). □

We prove (1.2) by proving a sequence of lemmas. We first extend the concept of a tail to a general graph. Let u and v be vertices of a graph G and let P be an induced uv-path. We call P a *tail* if all edges of G that are between $V(P)$ and $V(G) - V(P)$ are incident with u. We also call v the *tip* of the tail. Clearly, the tip v must have degree one, if $u \neq v$. Since u is not required to be adjacent with all vertices in $V(G) - V(P)$, the graph G does not have to be a tadpole graph.

2.3. *Suppose a graph G has a tail P of length at least two. If the non-tip end of P has degree greater than $r(n-1)$, where $n \geqslant 2$ is an integer, then G contains either $K^s_{1,n}$ or $\overline{K_{2,n}\backslash e}$ as an induced subgraph.*

Proof. Let x be the non-tip end of P. Let y be the unique neighbor of x in P and let z be the only other neighbor of y in P. Let N be the set of neighbors of x that are not in P. Clearly, $|N| \geqslant r(n-1)$. By applying Ramsey's Theorem to $G[N]$, we conclude that N has a subset X such that $G[X]$ is either K_{n-1} or $\overline{K_{n-1}}$. It follows that $G[X \cup \{x,y,z\}]$ is either $\overline{K_{2,n}\backslash e}$ or $K^s_{1,n}$. The lemma is proved. □

2.4. *Let G be a connected graph with more than $p_{n-1}(r(n-1))$ vertices, where $n \geqslant 2$ is an integer. If G has a tail P of length two, then G contains P_n, $K^s_{1,n}$ or $\overline{K_{2,n}\backslash e}$ as an induced subgraph.*

Proof. Let v be the tip of P. By (2.1), G has an induced tadpole graph H, with v as the tip of its tail, and such that either its tail has more than $n-1$ vertices or its body has more than $r(n-1)$ vertices. In the first case, H, and thus G, has an induced P_n. In the second case, the tail of H has length at least two, as v is the tip of P, which has length two. By applying (2.3) to H, we conclude that H, and hence G, contains an induced $K^s_{1,n}$ or $\overline{K_{2,n}\backslash e}$. The proof is completed. □

2.5. *Let xy be an edge of a connected graph G such that x has degree one in G and y has degree one in $\bar{G}$. Suppose $|V(G)| > 3r(n)$, where $n \geqslant 2$ is an integer. Then G contains $K^s_{1,n}$, $K_{2,n}\backslash e$, $K^+_{2,n}$, or the complement of one of these graphs, as an induced subgraph.*

Proof. Let z be the unique neighbor of y in $\bar{G}$ and let $X = V(G) - \{x,y,z\}$. Notice that $|X| = |V(G)| - 3 \geqslant 3r - 2$, where $r = r(n)$. Let Y be the set of vertices in X that are not incident with z in G. If $|X - Y| \geqslant r$, then, by applying (2.3) to the complement of $G - Y$, we conclude that G has $\overline{K^s_{1,n}}$ or $K_{2,n}\backslash e$ as an induced subgraph. Thus we may assume that $|X - Y| < r$, and so $|Y| \geqslant 2r - 1$. Since G is connected, some vertex, say u, must belong to $X - Y$. Let Z be the set of vertices in Y that are adjacent with u in G. If $|Y - Z| \geqslant r$, then, by applying (2.3) to $G[\{u,y,z\} \cup (Y - Z)]$, we conclude that G has $K^s_{1,n}$ or $\overline{K_{2,n}\backslash e}$ as an induced subgraph. Therefore, we may further assume that $|Y - Z| < r$, and so $|Z| \geqslant r$. Now, by applying Ramsey's Theorem to $G[Z]$, we conclude that Z has a subset U such that $G[U]$ is either K_n or $\bar{K}_n$. It follows that $G[\{x,y,z,u\} \cup U]$ is either $\overline{K^+_{2,n}}$ or $K^+_{2,n}$. The lemma is proved. □

2.6. *Let xy be an edge of a connected graph G such that x has degree one and y has degree less than $|V(G)| - 1$. Suppose $|V(G)| > p_{n-1}(3r(n) - 3)$, where $n \geqslant 2$ is an integer. Then G contains a graph in $\mathscr{U}_n$ as an induced subgraph.*

Proof. By (2.1), G contains an induced tadpole graph H with x as the tip of its tail and such that either its tail has more than $n-1$ vertices or its body has more than $3r(n) - 3$ vertices. In the first case, G contains an induced P_n. Thus we may assume that the body of H has at least $3r(n)-2$ vertices. If the tail of H has length at least two, then, by applying (2.3) to H, we conclude that H, and so G, contains an induced $K^s_{1,n}$ or $\overline{K_{2,n}\backslash e}$. Therefore, we only need to consider the case when the tail of H is the single edge xy. Clearly, it follows that y has degree greater than $3r(n)-2$.

Let X be the set of neighbors of y. Since G is connected and y has degree less than $|V(G)|-1$, there is a vertex, say z, such that $z \notin X \cup \{y\}$ and z is adjacent with at least one vertex in X. It is easy to see that $G[X \cup \{y,z\}]$ satisfies the assumptions in (2.5). Therefore, we conclude, in this and all the above cases, that G contains a graph in $\mathscr{U}_n$ as an induced subgraph. □

2.7. *Let x be a vertex of a doubly connected graph G such that every vertex is within distance two from x. If the degree of x is at least $2p_{n-1}(3r(n))-3$, where $n \geqslant 2$ is an integer, then G contains a graph in $\mathscr{U}_n$ as an induced subgraph.*

Proof. Without loss of generality, let us assume that G is a minimal graph, under the induced subgraph relation, that satisfies all the assumptions in (2.7). For $i=1,2$, let X_i be the set of vertices that are distance i away from x. Then $V(G)$ is the disjoint union of $\{x\}$, X_1, and X_2. Since $\bar{G}$ is connected, X_2 is not empty and thus we can choose a vertex y from X_2. By the minimality of G, it is clear that $\bar{G}-y$ must be disconnected. Let C be a component of $\bar{G}-y$ that has the least number of vertices. Then $\bar{G}-V(C)$ has at least

$$\frac{1}{2}(|V(G)|-1)+1 \geqslant p_{n-1}(3r(n))$$

vertices.

If y is not adjacent with a vertex $z \in V(C)$ in $\bar{G}$, then we choose an induced yz-path P in $\bar{G}$. Let H be obtained from $\bar{G}$ by deleting all vertices in $V(C)-V(P)$. It is easy to see that H is connected, P is a tail of H, and H satisfies the assumptions in (2.4). Therefore, H, and thus G, contains a graph in $\mathscr{U}_n$ as an induced subgraph. Next, we assume that, in $\bar{G}$, y is adjacent to all vertices of C. Let $z \in V(C)$ and let $H=\bar{G}-(V(C)-\{z\})$. Notice that H is connected. In addition, in H, yz is an edge, z has degree one, and y has degree less than $|V(H)|-1$, as y cannot be adjacent with all other vertices in $\bar{G}$. Now the result follows from (2.6). □

With the above preparations, now we are ready to prove our first main result, (1.2).

Proof of (1.2). Let $h(1)=1$, and for $n \geqslant 2$, let $h(n)=p_{n-1}(2p_{n-1}(3r(n)))+1$. The result is clear when $n=1$. Thus we assume in the following that $n \geqslant 2$. Let G be a doubly connected graph on at least $h(n)$ vertices. We need to show that some member of $\mathscr{U}_n$ is an induced subgraph of G.

By (2.2), we may assume that G has a vertex x of degree greater than $2p_{n-1}(3r(n))$, for otherwise G would contain an induced P_n, which is a member of $\mathscr{U}_n$. By (2.7), we may further assume that some vertex y is distance three away from x. Let X be the set of neighbors of x and let z be a neighbor of y such that z is adjacent to a vertex in X. Let $H=G[X \cup \{x,y,z\}]$. Observe that $|V(H)|>|X|>p_{n-1}(3r(n))$, $yz \in E(H)$,

y has degree one in H, and z has degree less than $|V(H)|-1$ in H, as $zx \notin E(H)$. Therefore, by applying (2.6) to H, we complete our proof of (1.2). □

For the sake of completeness, we also include a proof of (1.3).

Proof of (1.3). Notice that, by the definition of the closeness, the intersection of any family of closed classes remains to be a closed class. Therefore, $\mathscr{H}$, the smallest closed class that contains $\mathscr{G}$, does exist. From the definition of $\mathscr{G}^*$ it is not difficult to verify that $\mathscr{G}^*$ is a closed class that contains $\mathscr{G}$, and every closed class that contains $\mathscr{G}$ must also contains $\mathscr{G}^*$. Clearly, the first part of the last observation implies $\mathscr{H} \subseteq \mathscr{G}^*$, while the second part implies $\mathscr{H} \supseteq \mathscr{G}^*$. Thus $\mathscr{H} = \mathscr{G}^*$ is proved. □

We prove (1.4) by proving the following.

2.8. *The two statements* (1.2) *and* (1.4) *are equivalent.*

Proof. We first prove that (1.4) implies (1.2). Suppose the function h claimed in (1.2) does not exist. Then there exists a positive integer n such that $h(n)$ cannot be defined. What it means is that, for any positive integer k, there exists a doubly connected graph G_k on more than k vertices such that G_k does not contain any graph in $\mathscr{U}_n$ as an induced subgraph. However, by (1.4), $\mathscr{G}_n$ can be expressed as $\mathscr{G}^*$ for a finite $\mathscr{G}$. It follows that every G_k can be constructed, starting from graphs in $\mathscr{G}$, by repeatedly taking disjoint unions and taking complements. Since G_k is doubly connected, neither G_k nor $\bar{G}_k$ is the disjoint union of any two graphs. Therefore, at least one of G_k and $\bar{G}_k$ must be contained in $\mathscr{G}$. It follows that $\mathscr{G}$ is infinite, and this contradiction proves (1.2).

Next we prove that (1.2) implies (1.4). Let n be a positive integer and let $\mathscr{G}$ be the set of graphs in $\mathscr{G}_n$ that have fewer than $h(n)$ vertices, where $h(n)$ is the function defined in (1.2). Then it is enough for us to show that $\mathscr{G}_n = \mathscr{G}^*$. By the definition of $\mathscr{G}_n$, it is clear that $\mathscr{G}^* \subseteq \mathscr{G}_n$. Thus we only need to show that $\mathscr{G}_n \subseteq \mathscr{G}^*$. Suppose otherwise. Then we can choose a graph G in $\mathscr{G}_n - \mathscr{G}^*$ with the least number of vertices. From the definition of $\mathscr{G}$ we know that $|V(G)| \geqslant h(n)$. We also know, by (1.2), that G is not doubly connected. It follows that either G or $\bar{G}$ is the disjoint union of two smaller graphs G_1 and G_2. Since both G and $\bar{G}$ are in $\mathscr{G}_n$, both G_1 and G_2 are in $\mathscr{G}_n$. Now we deduce from the minimality of G that both G_1 and G_2 are in $\mathscr{G}^*$, which implies that their disjoint union is in $\mathscr{G}^*$. Consequently, both G and $\bar{G}$ are in $\mathscr{G}^*$. This contradiction proves (1.4). □

3. The first application

To prove (1.5), our second main result, we need some definitions. A binary relation $\leqslant$ on a set Q is a *quasi order* if $\leqslant$ is reflexive and transitive. It is a *well quasi order* (or a *wqo*) if, for every infinite sequence $q_1, q_2, \ldots$ of members of Q, there exist indices i and j such that $i < j$ and $q_i \leqslant q_j$. In this section, we denote by $\preccurlyeq$ the induced subgraph relation. That is, we write $G \preccurlyeq G'$ if G is isomorphic to an induced subgraph of G'. Clearly, $\preccurlyeq$ is a quasi order on the class of all graphs. However, $\preccurlyeq$ is not a wqo, as shown by the sequence $C_3, C_4, \ldots$, where C_n is the cycle on n vertices. In this section, we prove the following result and we show that it implies (1.5).

3.1. *For each positive integer n, graphs in $\mathscr{G}_n$ are well quasi ordered by $\preccurlyeq$.*

Let Π be a set of commutative and associative binary graph operations with the additional properties that:

(1) if $G \preccurlyeq G'$, $H \preccurlyeq H'$, and $\pi \in \Pi$, then $\pi(G,H) \preccurlyeq \pi(G',H')$; and
(2) if $\pi \in \Pi$, then $G \preccurlyeq \pi(G,H)$ and $H \preccurlyeq \pi(G,H)$, for all graphs G and H.

Then, for any π and π' in Π, we define

$$\pi \leqslant \pi' \text{ if } \pi(G,H) \preccurlyeq \pi'(G,H) \text{ for all graphs } G \text{ and } H.$$

For each class $\mathscr{G}$ of graphs, we also define $\mathscr{G}(\Pi)$ to be the class of all graphs constructed, starting from graphs in $\mathscr{G}$, by repeatedly using operations in Π. We will use the following result from [3] to prove (3.1).

3.2. *If both $(\mathscr{G}, \preccurlyeq)$ and $(\Pi, \leqslant)$ are well quasi orders, then so is $(\mathscr{G}(\Pi), \preccurlyeq)$.*

Proof of (3.1). We consider two graph operations. Let $\pi_1(G,H)$ be the disjoint union of G and H; let $\pi_2(G,H)$ be the complement of $\pi_1(\bar{G},\bar{H})$. Equivalently, $\pi_2(G,H)$ is obtained from the disjoint union of G and H by adding all edges between $V(G)$ and $V(H)$. It is obvious that each π_i is both commutative and associative. Let $\Pi = \{\pi_1, \pi_2\}$. Then $(\Pi, \leqslant)$ is a wqo since Π is finite.

Let $\mathscr{G}$ be the finite class of graphs determined in (1.4). Let $\bar{\mathscr{G}} = \{\bar{G} : G \in \mathscr{G}\}$ and let $\mathscr{H} = \mathscr{G} \cup \bar{\mathscr{G}}$. Then $\mathscr{G}^* \subseteq (\mathscr{G} \cup \bar{\mathscr{G}})^* \subseteq (\mathscr{G}^* \cup \bar{\mathscr{G}})^* = (\mathscr{G}^*)^* = \mathscr{G}^*$, which implies that $\mathscr{H}^* = \mathscr{G}^* = \mathscr{G}_n$. Since $\mathscr{G}$ is finite, it follows that $\mathscr{H}$ is finite, and thus $(\mathscr{H}, \preccurlyeq)$ is a wqo. Therefore, by (3.2), in order to prove (3.1), we only need to show that $\mathscr{H}(\Pi) = \mathscr{H}^*$.

Notice that each π_i can be expressed in terms of taking disjoint unions and taking complements. Thus $\mathscr{H}(\Pi) \subseteq \mathscr{H}^*$. On the other hand, we claim that $\mathscr{H}^* - \mathscr{H}(\Pi)$ is empty. Suppose otherwise. Then we can choose a graph G in $\mathscr{H}^* - \mathscr{H}(\Pi)$ with the least number of vertices. Since $\mathscr{H} \subseteq \mathscr{H}(\Pi)$, our graph G does not belong to $\mathscr{H}$. It follows from the definition of $\mathscr{H}$ that $\bar{G}$ does not belong to $\mathscr{H}$ either. Therefore, either G or $\bar{G}$ is the disjoint union of two smaller graphs, say G_1 and G_2, in $\mathscr{H}^*$. Equivalently, either $G = \pi_1(G_1, G_2)$ or $G = \pi_2(\bar{G}_1, \bar{G}_2)$. By the definition of $\mathscr{H}^*$, it is clear that both $\bar{G}_1$ and $\bar{G}_2$ belong to $\mathscr{H}^*$. Now we conclude from the minimality of G that G_1, G_2, $\bar{G}_1$, and $\bar{G}_2$ must all belong to $\mathscr{H}(\Pi)$. Consequently, our chosen graph G, which is either $\pi_1(G_1, G_2)$ or $\pi_2(\bar{G}_1, \bar{G}_2)$, must belong to $\mathscr{H}(\Pi)$. This is a contradiction, which completes the proof of (3.1). □

Proof of (1.5). Let $\mathscr{U}$ be the set of graphs G for which G does not have property **P** but all its proper induced subgraphs do. Since **P** is hereditary, it is easy to verify that a graph has property **P** if and only if it does not contain any graph in $\mathscr{U}$ as an induced subgraph. We need to show that $\mathscr{U}$ is finite.

From the definition of $\mathscr{U}$ it is easy to see that any two of its distinct members are incomparable under $\preccurlyeq$. Let $\mathscr{A}$ be the set of members G of $\mathscr{U}$ such that G is an induced subgraph of some graph in $\mathscr{U}_n$. We first claim that all graphs in $\mathscr{U} - \mathscr{A}$ are contained in $\mathscr{G}_n$. Suppose, on the contrary, that there exists a graph $G \in \mathscr{U} - \mathscr{A} - \mathscr{G}_n$. Then G contains a graph H in $\mathscr{U}_n$ as an induced subgraph. By the assumption of (1.5), H does not have property **P**. Thus, H contains a graph in $\mathscr{A}$ as an induced subgraph. It follows that G contains a graph in $\mathscr{A}$ as an induced subgraph. This is impossible, as no two distinct graphs in $\mathscr{U}$ are comparable under $\preccurlyeq$, so our claim is proved.

Clearly, $\mathscr{A}$ is finite. If $\mathscr{U} - \mathscr{A}$ were infinite, then there would exist an infinite sequence $G_1, G_2, \ldots$ of distinct graphs in $\mathscr{U} - \mathscr{A} \subseteq \mathscr{G}_n$. However, by (3.1), there would exist indices $i < j$ with $G_i \preccurlyeq G_j$. This is certainly impossible, as no two distinct graphs in $\mathscr{U}$ are comparable. Therefore, we conclude that $\mathscr{U} - \mathscr{A}$, and thus $\mathscr{U}$, is finite. □

4. The second application

We first prove (1.9), which is an easy corollary of (1.4).

Proof of (1.9). Suppose $\mathscr{H} \subseteq \mathscr{G}_n$, for some positive integer n. Then, by (1.4), $\mathscr{H} \subseteq \mathscr{G}^*$, for some finite $\mathscr{G}$. Conversely, suppose $\mathscr{H} \subseteq \mathscr{G}^*$, for some finite $\mathscr{G}$. Let n be an integer such that every graph in $\mathscr{G}$ has fewer than n vertices. Then no graph in $\mathscr{G}$ contains any graph in $\mathscr{U}_n$ as an induced subgraph. That is, $\mathscr{G} \subseteq \mathscr{G}_n$. Consequently, $\mathscr{G}^* \subseteq \mathscr{G}_n$, and thus $\mathscr{H} \subseteq \mathscr{G}_n$, as required. □

In the rest of the paper, we prove (1.10) and (1.11). We need the following well known fact on well quasi orders, which can be found, for instance, in [11].

4.1. *Let $(Q, \leqslant)$ be a wqo and let R be an infinite subset of Q. Then R contains an infinite sequence $r_1, r_2, \ldots$ such that $r_i < r_{i+1}$, for all i.*

Proof of (1.10) and (1.11). We prove (1.10) and (1.11) together. First notice that the three conditions in (1.10) obviously imply the three conditions in (1.11). Thus, we only need to prove the "only if" part of (1.10) and the "if" part of (1.11).

To prove the "only if" part of (1.10), let $\mathscr{H} = \mathscr{G}^*$, for some finite $\mathscr{G}$. Then (1) follows from (1.3) and (2) follows from (1.9). Now we prove (3). Since $\mathscr{S}$ is infinite and $\mathscr{G}$ is finite, we can take a graph G in $\mathscr{S}$ such that every graph in $\mathscr{G}$ has fewer than $|V(G)|$ vertices. Clearly, neither G nor $\bar{G}$ is in $\mathscr{G}$. Therefore, one of these two graphs has to be the disjoint union of two other graphs in $\mathscr{G}^*$. By our assumption on $\mathscr{S}$, $\mathscr{G}$ is disconnected, and thus $\bar{G}$ is connected. It follows that G is the disjoint union of two other graphs in $\mathscr{G}^*$, and so (3) is proved.

Next, we prove the "if" part of (1.11). Let us call a graph G in $\mathscr{H}$ *essential* if neither G nor $\bar{G}$ is the disjoint union of two other graphs in $\mathscr{H}$. Let $\mathscr{G}$ be the class of all essential graphs in $\mathscr{H}$. Notice that every graph in $\mathscr{H}$ can be constructed, starting from graphs in $\mathscr{G}$, by repeatedly taking disjoint unions and taking complements, so $\mathscr{H} \subseteq \mathscr{G}^*$. On the other hand, since $\mathscr{H}$ is closed, by (1), and $\mathscr{H} \supseteq \mathscr{G}$, it follows from (1.3) that $\mathscr{H} \supseteq \mathscr{G}^*$. Therefore, we have $\mathscr{H} = \mathscr{G}^*$. Now it remains to show that $\mathscr{G}$ is finite.

Suppose, on the contrary, that $\mathscr{G}$ is infinite. It follows from (2) and (1.4) that $\mathscr{G}$ contains an infinite set $\mathscr{S}$ of graphs G such that either G or $\bar{G}$ is disconnected. From the definition of $\mathscr{G}$ we deduced that a graph is in $\mathscr{G}$ if and only if its complement is in $\mathscr{G}$. Therefore, without loss of generality, we may assume that all graphs in $\mathscr{S}$ are disconnected. Consequently, by (3.1) and (4.1), there exists an infinite monotone sequence $G_1, G_2, \ldots$ such that every G_i is in $\mathscr{S}$. Now, by (3), some G_i is the disjoint union of two other graphs in $\mathscr{H}$, contradicting the definition of $\mathscr{G}$. This contradiction completes our proof. □

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