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Generating r-regular graphs ☆

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Abstract

For each nonnegative integer r, we determine a set of graph operations such that all r-regular loopless graphs can be generated from the smallest r-regular loopless graphs by using these operations. We also discuss possible extensions of this result to r-regular graphs of girth at least g, for each fixed g.

Keywords: Graph generation; Regular graph

1. Introduction

A well-known classical theorem of Steinitz and Rademacher [22] states that the class $\mathscr{G}$ of 3-connected 3-regular planar simple graphs can be generated from the Tetrahedron by *adding handles*, a graph operation illustrated in Fig. 1 below.

This result can be stated more precisely as follows. For every graph G in $\mathscr{G}$, there is a sequence $G_0, G_1, \ldots, G_t$ of members of $\mathscr{G}$ such that G_0 is the Tetrahedron, G_t is G, and each G_i, where $1 \leqslant i \leqslant t$, is obtained from G_{i-1} by adding a handle. In [1–3,6,8–12,15,21,23,24,26] and [27], analogous results are obtained for various other families of 3-regular simple graphs. For instance, in [8] and [12], it is proved that the class of cyclically 4-connected 3-regular planar graphs can be generated from the Cube by adding handles. For 4-regular simple graphs, the situation is similar and the readers are referred to [4,5,13,14,16–20,25]. In this paper, we will consider the general problem. of generating r-regular (not necessarily simple) graphs, for each fixed r.

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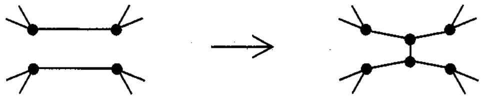


Fig. 1. Adding a handle.

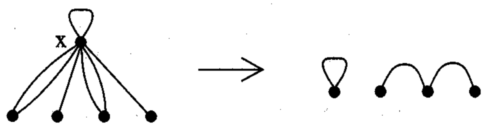


Fig. 2. Splitting a vertex x.

As a matter of fact, instead of trying to generate all r-regular graphs, we will consider how to reduce an r-regular graph to a smaller r-regular graph. This is an equivalent problem but it is more convenient to work with. To be more precise, let $\mathscr{G}$ be a class of graphs. We say that a graph $G \in \mathscr{G}$ can be *reduced* within $\mathscr{G}$ by a set $\mathscr{O}$ of operations to a graph $H \in \mathscr{G}$ if there is a sequence $G_0, G_1, \ldots, G_t$ of members of $\mathscr{G}$ such that $G_0 = G$, $G_t = H$, and each G_i, where $1 \leqslant i \leqslant t$, is obtained from G_{i-1} by applying an operation in $\mathscr{O}$ only once.

We first define an operation that we are going to use in this paper. Let x be a vertex of a graph G and let $\{e_i\colon\, i = 1, 2, \ldots, m\}$ be the set of non-loop edges that are incident with x. If x has an even degree and $e_i = xx_i$, for all i, then the result of *splitting* x (see Fig. 2) is a graph obtained from $G - x$ by adding $m/2$ new edges $x_1x_2, x_3x_4, \ldots, x_{m-1}x_m$. When $m > 2$, it is clear that, depending on how the non-loop edges are paired, there are different ways to split x.

Observe that, splitting a vertex does not change the degree of any other vertex in the graph. In particular, when r is even, the result of splitting a vertex in an r-regular graph remains being r-regular. Therefore, if $\mathscr{G}$ is the class of all r-regular graphs, where r is even, then every graph in $\mathscr{G}$ can be reduced within the class to the graph with one vertex and $r/2$ loops by splitting vertices. Equivalently, we can say that, when r is even, every r-regular graph can be constructed from the unique r-regular graph on one vertex by the following operation (the reverse operation of splitting a vertex): Delete any $p \leqslant r/2$ distinct edges, say $x_1x_2, x_3x_4, \ldots, x_{2p-1}x_{2p}$, from the given graph, add a new vertex x, add $r/2 - p$ loops to x, and also add all edges in $\{xx_i\colon\, i = 1, 2, \ldots, 2p\}$. Similarly, if $\mathscr{G}$ is the class of all r-regular graphs, where r is odd, then every graph in $\mathscr{G}$ can be reduced within $\mathscr{G}$ to one of the $(r+1)/2$ r-regular graphs on two vertices by the following operation: Delete a non-loop edge xy from the given graph and then split both x and y, in any order.

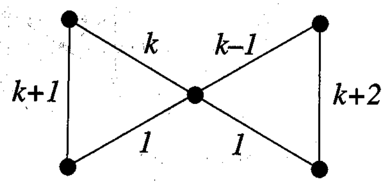


Fig. 3. Graph Γ.

From the above discussion one can see that, if loops are allowed, then the problem of generating r-regular graphs is easy. Therefore, we will concentrate on loopless graphs. For each positive integer r, let $\mathscr{G}_r$ be the class of all loopless r-regular graphs. Let us denote by S the operation of splitting vertices. We point out that, when r is even, there are many r-regular graphs that cannot be reduced within $\mathscr{G}_r$ by S. To see this, take any graph G in $\mathscr{G}_{\frac{r}{2}}$ such that G has a perfect matching M. For each edge in M, add $r/2$ edges parallel to it. Then we end up with a graph G' in $\mathscr{G}_r$. Now it is straightforward to verify that splitting any vertex of G' will result loops. This observation suggests that, in order to reduce all even regular loopless graphs, another operation is necessary.

Let e be an edge of a graph G in $\mathscr{G}_r$. We will call e *heavy* if there are at least $(r-1)/2$ other edges that are parallel with e. Equivalently, the parallel family that contains e contains more than $r/2$ edges. If r is even and $e=xy$ is heavy, then a *double split* at e is the operation (denoted by DS) of splitting both x and y, in any order. Clearly, when e is heavy, the result of splitting any one of x and y must have loops. However, it is very possible that splitting both x and y, that is, a double split at e, may result a graph in $\mathscr{G}_r$. The next is our first main result. For each positive integer p, let pK_2 be the graph with two vertices and p parallel edges.

Theorem 1. *If r is a positive even integer, then every graph in $\mathscr{G}_r$ can be reduced within $\mathscr{G}_r$ to rK_2 by* $\{S,\mathrm{DS}\}$.

For odd regular graphs, the natural operation is the one we mentioned earlier: Delete an edge xy and then split both x and y, in any order. We denote this operation by DS^+.

Theorem 2. *Every graph in $\mathscr{G}_3$ can be reduced within $\mathscr{G}_3$ to $3K_2$ by* DS^+.

This result could have been discovered before, but we cannot find a reference. For completeness, we include a proof of this result in this paper.

For odd r exceeding three, the situation is different. We point out that, similar to the case for even regular graphs, the operation DS^+ alone is not enough to reduce all graphs in $\mathscr{G}_r$. To see this, consider the graph Γ illustrated in Fig. 3, where $k=(r-1)/2$, and the label next to each edge indicates the size the corresponding parallel family. Notice that the degrees of the five vertices are $k+2$, $k+3$, $2k+1$, $2k+1$, and $2k+1$, respectively.

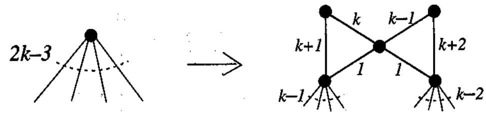


Fig. 4. Modifying each vertex of H.

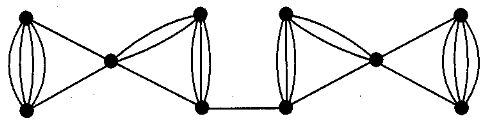


Fig. 5. Graph Ψ.

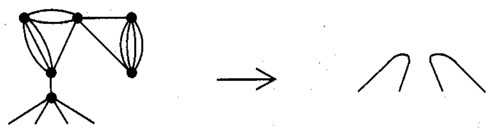


Fig. 6. Operation R.

Take any $(2k-3)$-regular loopless graph H and modify each of its vertices as illustrated in Fig. 4. That is, at each vertex, partition the $2k-3$ neighboring edges into two groups, one of size $k-1$ and one of size $k-2$, and then attach each group of edges to the corresponding vertex in a copy of Γ. Clearly, the resulting graph G is loopless and r-regular. It is straightforward to verify that applying DS^+ to any edge of G must create at least one loop.

In particular, when $r=5$, it is clear that every component of H is K_2. Suppose H has p components. Then the above modified graph G also has p components, each of which is isomorphic to the graph Ψ illustrated in Fig. 5. We will refer this graph G as Ψ^p.

Theorem 3. *Every graph in $\mathscr{G}_5$ can be reduced within $\mathscr{G}_5$ to $5K_2$ or Ψ^p, for some p, by $\{\mathrm{DS}^+, R\}$, where R is the operation illustrated in* Fig. 6.

Just like DS is the result of applying S twice, it is not difficult to see that operation R can be realized by applying DS^+ three times. If we insist on using operations of this kind, i.e. repeatedly applying DS^+ several times, it seems quite unlikely that there is a set of operations, like in Theorem 1, that works for all odd r.

To deal with general odd regular loopless graphs, we need to introduce a different operation. This is an analog of the operation defined in [10,21], which was also studied in [25]. Let $e=xy$ be an edge in a graph $G\in\mathscr{G}_r$, where r is odd. The new operation, which will be denoted by DS^-, consists of two steps when it is applied to e. We first contract all, say k, edges between x and y. That is, we delete all these k edges and also identify x with y. Clearly, the new vertex has an even degree $2r-2k$. Thus we can split this new vertex, which is the second step of our operation.

Theorem 4. *If r is odd, then every graph in $\mathscr{G}_r$ can be reduced within $\mathscr{G}_r$ to rK_2 by* DS$^-$.

The next question is: how do we generate regular simple graphs, or in general, regular graphs of girth at least g? We can do it by modifying the known results if we are allowed to relax a little on the generating procedure. More discussion on this is given in the last section of this paper.

2. Even regular graphs

In this section, r is a positive even integer. To prove Theorem 1, we first prove two lemmas. For any two vertices x and y of a graph G, let $\mu_G(x,y)$ be the number of edges of G that are between x and y.

Lemma 2.1. *Let x be a vertex of a loopless graph G, which has at least two vertices. Suppose x has an even degree, say d, and $\mu_G(x,y)\leqslant d/2$, for all y. Then x can be split to result a loopless graph.*

Proof. We prove the lemma by induction on d. Since the result is obviously true when $d=0$, we may assume that $d>0$. Notice that x has at least two neighboring vertices, as $\mu_G(x,y)\leqslant d/2<d$, for all y. Thus we can choose distinct vertices y_1 and y_2, other than x, such that $\mu_G(x,y_1)\geqslant\mu_G(x,y_2)\geqslant\mu_G(x,y)$, for all $y\neq y_1$. Let G' be obtained from G by deleting two edges xy_1 and xy_2, and also adding an edge y_1y_2. Clearly, G' is loopless and x has degree $d-2$ in G'. We claim that $\mu_{G'}(x,y)\leqslant(d-2)/2$, for all y. Suppose, on the contrary, that $\mu_{G'}(x,y_0)>(d-2)/2$, for some y_0. Then it is obvious that $y_0\notin\{x,y_1,y_2\}$, as $\mu_{G'}(x,x)=0$, and $\mu_{G'}(x,y_i)=\mu_G(x,y_i)-1<d/2$, for $i=1,2$. Therefore, $\mu_G(x,y_1)\geqslant\mu_G(x,y_2)\geqslant\mu_G(x,y_0)\geqslant\mu_{G'}(x,y_0)\geqslant d/2$, and it follows that $d\geqslant\mu_G(x,y_1)+\mu_G(x,y_2)+\mu_G(x,y_0)\geqslant 3d/2$, a contradiction, which proves the claim. Now, by induction, we can split x in G' to obtain a loopless graph. Consequently, by the definition of G', we can split x in G to obtain a loopless graph. □

For any three distinct vertices x, y, and z of a graph G, let us define $\mu_G(x,y,z)$ to be $\mu_G(x,y)+\mu_G(y,z)+\mu_G(z,x)$.

Lemma 2.2. *Let $G\in\mathscr{G}_r$ have at least three vertices. If $e=xy$ is a heavy edge in G and $\mu_G(x,y,z)\leqslant r$ for all $z\neq x,y$. Then the operation* DS *can be applied to e to result a loopless graph.*

Proof. Since G has more than two vertices, we only need to exhibit a way of splitting x and y such that the resulting graph is loopless. Let $\mu=\mu_G(x,y)$. Then $\mu>r/2$, as e is heavy. Let G_1 be obtained from $G-y$ by adding $\mu-r/2$ loops to x, and also adding $\mu_G(y,z)$ new edges between x and z, for all $z \neq x, y$. It is easy to see that G_1 is an r-regular graph obtained from G by splitting y. In addition, $\mu_{G_1}(x,z) = \mu_G(x,z) + \mu_G(y,z)$, for all $z \neq x, y$. Let G_1' be obtained from G_1 by deleting its $\mu - \frac{r}{2}$ loops at x. Then G_1' is loopless and x has degree $d = r - 2(\mu - \frac{r}{2}) = 2r - 2\mu$ in G_1'. Moreover, as $\mu_G(x,y,z) \leqslant r$, we have $\mu_{G_1'}(x,z) = \mu_{G_1}(x,z) \leqslant r - \mu = d/2$. Now, by Lemma 2.1, we can split x in G_1' to result a loopless graph G_2. From the definition of G_1' it is clear that G_2 is also a result of splitting x in G_1. Therefore, the lemma is proved. □

Proof of Theorem 1. Clearly, we only need to show that, if $G \in \mathscr{G}_r$ has three or more vertices, then at least one of S and DS can be applied to result a smaller graph in $\mathscr{G}_r$. By Lemma 2.1, we may assume that every vertex is incident with a heavy edge. Let

$$\mu = \min\{\mu_G(x,y)\colon xy \text{ is a heavy edge of } G\}$$

and let $e = x_1y_1$ be an edge with $\mu_G(x_1,y_1) = \mu$. We claim that $\mu_G(x_1,y_1,z) \leqslant r$ for all $z \neq x_1, y_1$. Suppose, on the contrary, that $\mu_G(x_1,y_1,z_1) > r$ for some $z_1 \neq x_1, y_1$. Let $f = z_1u$ be a heavy edge incident with z_1. Then u is not x_1 or y_1, as any two incident heavy edges must be in parallel. It follows that

$$\begin{aligned}
\mu_G(z_1,u) &\leqslant r - \mu_G(z_1,x_1) - \mu_G(z_1,y_1)\\
&< \mu_G(x_1,y_1,z_1) - \mu_G(z_1,x_1) - \mu_G(z_1,y_1)\\
&= \mu_G(x_1,y_1)\\
&= \mu,
\end{aligned}$$

contradicting the definition of μ and thus our claim is proved. Now, by Lemma 2.2, we conclude that, in this case, DS can be applied to e to result a graph in $\mathscr{G}_r$. □

3. 3-regular and 5-regular graphs

We prove Theorems 2 and 3 in this section.

Proof of Theorem 2. Let G be a graph in $\mathscr{G}_3$ such that G has more than two vertices. We need to show that DS^+ can be applied to some edge to result a loopless graph. If G is simple, then it is clear that applying DS^+ to any edge of G will result a loopless graph. Thus we may assume that G has an edge $e = xy$ such that e is parallel to at least one other edge. If e is parallel to two other edges, then the component that contains e must have precisely two vertices and three edges. Notice that applying DS^+ to e is the same as deleting both x and y from G, which results a loopless graph. Thus we may assume that e is parallel to exactly one other edge. Let u_x be the only other neighboring vertex of x and u_y be the only other neighboring vertex of y. Observe that applying DS^+ to e is the same as deleting x and y, and then adding a new edge u_xu_y. Thus, if $u_x \neq u_y$, we can applying DS^+ to e and we are done. Now, suppose $u_x = u_y = u$. Clearly, u has a third neighboring vertex, say z. If z has three distinct neighboring vertices, then applying DS^+ to the edge uz will result a loopless graph.

Else, z has only one other neighboring vertex, say v, and such that $\mu_G(z,v)=2$. Let w be the other neighboring vertex of v. Notice that $w \neq u$. It follows that DS^+ can be applied to an edge between z and v to result a loopless graph. The theorem is proved. □

We prove Theorem 3 by proving a sequence of lemmas. If e is an edge of a graph G, then $G \setminus e$ is the graph obtained from G by deleting e.

Lemma 3.1. *Let $e=x_1x_2$ be an edge of $G \in \mathscr{G}_5$. Suppose both $\mu_G(x_i,y) \leqslant 2$ and $\mu_G(x_1,x_2,y) \leqslant 5$ hold for all $i \in \{1,2\}$ and all $y \in V(G)-\{x_1,x_2\}$. Then DS^+ can be applied to e to result a graph in $\mathscr{G}_5$, as long as $|V(G)|>2$.*

Proof. Let $G'=G\setminus e$. We first consider the case when some x_i is incident with a parallel family of size three or more in G'. Notice that such a family must be between x_1 and x_2, as $\mu_{G'}(x_i,y)=\mu_G(x_i,y) \leqslant 2$, for $i=1,2$ and $y \in V(G)-\{x_1,x_2\}$. If $\mu_{G'}(x_1,x_2)=4$, then applying DS^+ to e in G means deleting x_1 and x_2 from G, which obviously results a graph in $\mathscr{G}_5$, as $|V(G)|>2$. If $\mu_{G'}(x_1,x_2)=3$, then each x_i has exactly one other neighboring vertex, say y_i. Since $\mu_G(x_1,x_2,y) \leqslant 5$, for all $y \in V(G)-\{x_1,x_2\}$, we must have $y_1 \neq y_2$. It follows that applying DS^+ to e in G is the same as deleting vertices x_1,x_2 from G and then adding a new edge y_1y_2. Again, it is clear that the resulting graph is in $\mathscr{G}_5$.

Next, we assume that, in G', each parallel family that is incident with some x_i must have size at most two. Let us also assume, by renaming x_1 and x_2, if necessary, that, in G', either no parallel family of size two is incident with any x_i, or there is such a family that is incident with x_1. By Lemma 2.1, we can split x_1 to result a loopless graph, say G_1. We prove that

$$\mu_{G_1}(x_2,y) \leqslant 2, \quad \text{for all } y \in V(G_1)-\{x_2\}. \tag{$*$}$$

Suppose, on the contrary, that $\mu_{G_1}(x_2,y) \geqslant 3$, for some $y \in V(G_1)-\{x_2\}$. We consider two cases.

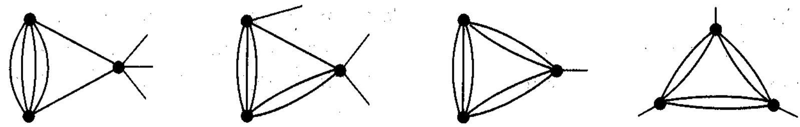
Case 1: At least two edges between x_2 and y in G_1 are not in G'. To produce these new edges, we must have $\mu_{G'}(x_1,x_2) \geqslant 2$ and $\mu_{G'}(x_1,y) \geqslant 2$. Since x_1 has degree four in G', we conclude that $\mu_{G'}(x_1,x_2)=\mu_{G'}(x_1,y)=2$, which in turn implies that $\mu_{G'}(x_2,y)=\mu_{G_1}(x_2,y)-2 \geqslant 1$. Therefore, we have $\mu_G(x_1,x_2,y)>5$, a contradiction.

Case 2: At most one edge between x_2 and y in G_1 is not in G'. In other words, $\mu_{G_1}(x_2,y)-\mu_{G'}(x_2,y) \leqslant 1$. Since $\mu_{G_1}(x_2,y) \geqslant 3$ and $\mu_{G'}(x_2,y) \leqslant 2$, it follows that

(i) $\mu_{G'}(x_2,y)=2$; and
(ii) $\mu_{G_1}(x_2,y)-\mu_{G'}(x_2,y)=1$.

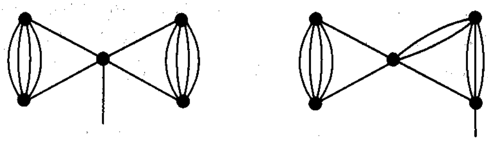
By (ii), G_1 has a new edge between x_2 and y, and thus we have $\mu_{G'}(x_1,x_2) \geqslant 1$ and $\mu_{G'}(x_1,y) \geqslant 1$. On the other hand, from (i) and $\mu_G(x_1,x_2,y) \leqslant 5$ we deduce that $\mu_{G'}(x_1,x_2)+\mu_{G'}(x_1,y) \leqslant 2$. Therefore,

(iii) $\mu_{G'}(x_1,x_2)=\mu_{G'}(x_1,y)=1$.



A_1 A_2 A_3 A_4

Fig. 7. There are only four kinds of heavy triangles.



A_{11} A_{12}

Fig. 8. How two heavy triangles meet.

Since, by (i), x_2 is incident with a parallel family of size two in G', the assumption we made before (*) implies that $\mu_{G'}(x_1,z)=2$, for some z. From (iii) it is clear that z is a vertex other than x_2 and y. Consequently, (ii) implies that the way we split x_1 creates a loop, which is a contradiction. This contradiction settles Case 2 and thus completes the proof of (*).

Now, Lemma 2.1 and (*) imply that we can split x_2 in G_1 to result a loopless graph. Thus the lemma is proved. □

Motivated by the last lemma, we call the subgraph induced by three distinct vertices x, y, and z a *heavy triangle* if $\mu_G(x,y,z)>5$.

Lemma 3.2. *The only heavy triangles are those illustrated in* Fig. 7.

Proof. Let T be a heavy triangle with vertices x_1, x_2, and x_3. If $\mu_G(x_i,x_j)\leqslant 2$, for all $i\neq j$, then $T=A_4$. If $\mu_G(x_i,x_j)\geqslant 4$, for some $i\neq j$, then $T=A_1$. The only case left is when $\mu_G(x_i,x_j)=3$, for some $i\neq j$. In this case, T must be A_2 or A_3. □

Lemma 3.3. *If two distinct heavy triangles have at least one vertex in common, then they must be as illustrated in* Fig. 8.

Proof. Let T be a heavy triangle with vertices x_1, x_2, and x_3. Then, by Lemma 3.2, for any $i\neq j$ and any vertex $y\in V(G)-V(T)$, the subgraph induced by $\{x_i,x_j,y\}$ has at most four edges. It follows that no two distinct heavy triangles can have two vertices in common. When two triangles have exactly one vertex in common, by Lemma 3.2 again, it is easy to see that none of them is A_3 or A_4, and they cannot be both A_2. Thus one of them is A_1 and the other is either A_1 or A_2. The lemma is proved. □

Let us call a graph in $\mathscr{G}_5$ *irreducible* if the application of DS^+ to any edge of the graph results at least one loop. The next lemma tells us the edge distribution of an irreducible graph G. Let E_1 be the set of all edges that are contained in a heavy triangle, and let $E_2 = E(G) - E_1$. Let X be the set of vertices that are the degree four vertex in a heavy triangle of type A_2.

Lemma 3.4. *Every edge in E_2 is incident with a vertex in X.*

Proof. Let $e = x_1x_2 \in E_2$. Since G is irreducible and e is not contained in any heavy triangle, by Lemma 3.1, we have $\mu_G(x_i, y) \geqslant 3$, for some $i = 1, 2$ and some $y \neq x_1, x_2$. Now by applying Lemma 3.1 to edge $f = x_iy$ we conclude that f is contained in a heavy triangle. It follows from Lemma 3.2 that $x_i \in X$ and thus the lemma is proved. □

Proof of Theorem 3. Let $G = (V, E)$ be a graph in $\mathscr{G}_5$. We need to show that, unless G is $5K_2$ or Ψ^p, for some p, at least one of DS^+ and R can be applied to G to result a graph in $\mathscr{G}_5$.

If $G = 5K_2$, we do not need to do anything. Thus we may assume that G has more than two vertices. We may also assume that G is irreducible. It follows that every component of G has more than two vertices, because otherwise, $5K_2$ is a component of G and DS^+ can be applied to an edge in this component to result a graph in $\mathscr{G}_5$, contradicting the assumption that G is irreducible. In fact, by considering each component, we may assume that G is connected and we only need to show that either $G = \Psi$ or operation R can be applied to result a graph in $\mathscr{G}_5$. Let E_1, E_2, and X be defined as in Lemma 3.4.

We observe, by Lemma 3.4, that G must have heavy triangles. We also observe that E_2 is not empty. This is clear, if G has a heavy triangle T that does not meet any other heavy triangles, as edges between $V(T)$ and $V - V(T)$ must belong to E_2. On the other hand, when G has two heavy triangles, say T_1 and T_2, that meet, then, by Lemma 3.3, there is a unique edge between $V(T_1) \cup V(T_2)$ and $V - (V(T_1) \cup V(T_2))$. It is clear that this edge must belong to E_2.

Let E_2' be the set of edges in E_2 for which both of its ends are contained in X. We first consider the case when $E_2' = E_2$. Let $e = x_1x_2 \in E_2'$. For $i = 1, 2$, let T_i be the heavy triangle of type A_2 that contains x_i as its degree four vertex, and let f_i be an edge that is not in $\{e\} \cup E(T_i)$ but is incident with a vertex of T_i. By the definition of E_2', we have $f_i \notin E_2' = E_2$ and thus $f_i \in E_1$. It follows that T_i meets another heavy triangle and so, by Lemma 3.3, that $G = \Psi$.

Next, we assume that $E_2'' = E_2 - E_2' \neq \emptyset$. To find a subgraph where operation R can be applied, we define a directed graph as follows. For each $e = xy \in E_2''$, by Lemma 3.4, exactly one of its ends, say x, is contained in X. Let us direct e from y to x. Then we delete all edges in E_2' and contract all edges in E_1. Let G^* be the resulting directed graph.

In every directed graph, since the sum of the outdegrees of the vertices equals the sum of the indegrees of the vertices, there must be a vertex for which its indegree is greater than or equal to its outdegree. Let v be such a vertex in G^*. Since $E_2'' \neq \emptyset$, we may choose v with an additional property that its indegree is greater than zero.

Notice that G^* has two kinds of vertices, those that are vertices of G and those that are created when contracting edges in E_1. It is easy to see that each vertex of the second kind corresponds to a component of G_1, the subgraph of G induced by edges in E_1. By Lemmas 3.2 and 3.3, these components are graphs in Figs. 7 and 8.

Let $a=uv$ be a directed (from u to v) edge in G^* and let $e=xy$ be the corresponding undirected edge in E_2''. By definition, precisely one end of e, say x, is contained in a heavy triangle, say T, of type A_2, as a degree four vertex. According to the way each edge in E_2'' is directed, we can see that v corresponds to x. Since x is contained in T and all edges of T are contracted, v is not x. Moreover, v is not the result of contracting $E(T)$, because otherwise, v would have indegree one and outdegree two in G^*, contradicting the choice of v. Therefore, v is the result of contracting a component C of G_1 of type A_{12}.

To complete our proof, it is enough to show that operation R can be applied to the component C. That is, by Lemma 2.1, we need to show that y is not incident in G with a parallel family of size three or more. Suppose, on the contrary, that $\mu_G(y,z) \geqslant 3$, for some z. Since G is irreducible, applying DS$^+$ to an edge $f=yz$ will result a loop. It follows from Lemma 3.1 that f is contained in a heavy triangle T'. By Lemma 3.2, T' is of type A_2 and y is the degree four vertex of T'. But this means that $e \in E_2'$, a contradiction. The theorem is proved. □

4. Odd regular graphs

In this section, r is a positive odd integer. We prove Theorem 4, like before, by proving a sequence of lemmas.

Lemma 4.1. *Let $G \in \mathcal{G}_r$ have more than two vertices. If $e=xy$ is an edge in G and $\mu_G(x,y,z) \leqslant r$ for all $z \neq x,y$. Then operation* DS$^-$ *can be applied to e to result a loopless graph.*

Proof. Let G' be obtained from G by contracting all edges between x and y. Let u be the new vertex in G'. Then u has degree $d = 2r - 2\mu_G(x,y)$. Moreover, for each vertex $z \in V(G') - \{u\}$, it is easy to see that

$$\begin{aligned}\mu_{G'}(u,z) &= \mu_G(x,z) + \mu_G(y,z)\\ &= \mu_G(x,y,z) - \mu_G(x,y)\\ &\leqslant r - \mu_G(x,y)\\ &= d/2.\end{aligned}$$

By Lemma 2.1, we conclude that u can be split to result a loopless graph. Therefore, DS$^-$ can be applied to e to result a loopless graph. □

Like in the last section, if $G \in \mathcal{G}_r$ and $\mu_G(x,y,z) > r$, then we call the subgraph induced by x, y, and z a *heavy triangle*. In this section, we do not need to distinguish different types of the heavy triangles, but it is worth noticing that in a heavy triangle there is at least one edge between each pair of vertices. Next, we study the distribution of the heavy triangles.

Lemma 4.2. *No two heavy triangles have exactly two vertices in common.*

Proof. Suppose, on the contrary, that there are two heavy triangles with vertex sets $\{x,y,u\}$ and $\{x,y,v\}$, respectively, and such that $u \neq v$. Then

$$\begin{aligned} 2r &< \mu_G(x,y,u) + \mu_G(x,y,v) \\ &= (\mu_G(x,y) + \mu_G(y,u) + \mu_G(u,x)) + (\mu_G(x,y) + \mu_G(y,v) + \mu_G(v,x)) \\ &= (\mu_G(x,y) + \mu_G(x,u) + \mu_G(x,v)) + (\mu_G(y,x) + \mu_G(y,u) + \mu_G(y,v)) \\ &\leqslant 2r, \end{aligned}$$

a contradiction. □

Now we define a bipartite graph H with vertex set $T \cup V(G)$, where $\mathscr{T}$ is the set of all heavy triangles, and such that $x \in V(G)$ is adjacent to $T \in \mathscr{T}$ in the new graph H if and only if $x \in V(T)$.

Lemma 4.3. *H is a forest.*

Proof. Suppose, on the contrary, that H has a cycle, say C. Let the vertices of C be $x_1, T_1, x_2, T_2, \ldots, x_p, T_p$. Let $F_i = E(T_i)$, for $i = 1,2,\ldots,p$, and let $F = F_1 \cup F_2 \cup \cdots \cup F_p$. By Lemma 4.2, it is clear that $|F| = |F_1| + |F_2| + \cdots + |F_p|$. For each $i = 1,2,\ldots,p$, let d_i be the number of edges in F that are incident with x_i. Since each T_i contains at least two vertices in $\{x_1,x_2,\ldots,x_p\}$, it follows that every edge in F is incident with at least one x_i, and thus $|F| \leqslant d_1 + d_2 + \cdots + d_p$. Consequently,

$$p \cdot r \geqslant d_1 + d_2 + \cdots + d_p \geqslant |F| = |F_1| + |F_2| + \cdots + |F_p| > p \cdot r$$

a contradiction. □

Let us call a sequence $T_1, T_2, \ldots, T_p$ of distinct heavy triangles *connected* if, for each $i = 2,3,\ldots,p$, there exists $j \in \{1,2,\ldots,i-1\}$ such that $V(T_i) \cap V(T_j) \neq \emptyset$.

Lemma 4.4. *If $p \geqslant 2$ and the sequence $T_1, T_2, \ldots, T_p$ of distinct heavy triangles is connected, then $|V_p \cap (V_1 \cup V_2 \cup \cdots \cup V_{p-1})| = 1$, where each V_i is $V(T_i)$.*

Proof. For each $i = 1,2,\ldots,p$, let $X_i = V_1 \cup V_2 \cup \cdots \cup V_i$ and let H_i be the subgraph of H induced by $X_i \cup \{T_1, T_2, \ldots, T_i\}$. Since the sequence $T_1, T_2, \ldots, T_p$ is connected, it is not difficult to see that each H_i is connected. Suppose $|V_p \cap X_{p-1}| \geqslant 2$. Then there are two distinct vertices, say x and y, that belong to both V_p and X_{p-1}. It follows that $xT_p \in E(H)$, $yT_p \in E(H)$, and H_{p-1} has a path, say P, between x and y. Consequently, H has a cycle $P \cup \{xT_p, yT_p\}$, contradicting Lemma 4.3. □

Proof of Theorem 4. Clearly, we only need to show that, for each $G \in \mathscr{G}_r$ with more than two vertices, DS^- can be applied to G to result a loopless graph. By Lemma 4.1, we may assume that G has at least one heavy triangle. Let $T_1, T_2, \ldots, T_p$ be a connected sequence of heavy triangles such that p is maximum. For $i = 1, 2, \ldots, p$, let $V_i = V(T_i)$ and let $X_i = V_1 \cup V_2 \cup \cdots \cup V_i$. Then, by Lemma 4.4, each X_i, where $2 \leqslant i \leqslant p$, has exactly two vertices more than X_{i-1}. Therefore, $|X_p| = 2p+1$, which is an odd number. As G is odd regular, there must exist an edge e for which precisely one of its ends is in X_p. We claim that there is no heavy triangle that contains e. Suppose that there exists such a heavy triangle T. Then $X_p \cap V(T) \neq \emptyset$ and $T \neq T_i$, for all i. It follows that the sequence $T_1, T_2, \ldots, T_p, T$ is connected, contradicting the maximality of p, and thus the claim is proved. Now, by Lemma 4.1 again, we conclude that the result of applying DS^- to e is a graph in $\mathscr{G}_r$. □

5. Regular graphs of large girth

Results in this paper are about loopless graph. A natural question is: what about simple graphs, or more generally, what about graphs of girth at least g? We do not intent to propose any conjecture on what kind of operations would work, since we do not know. What we are going to discuss here is the possibility of the existence of such operations. In order to make it clear, we need to introduce some definitions.

Let $\mathscr{G}$ be a class of graphs that we would like to generate. First we need to have a subclass, say $\mathscr{G}_0$, such that the rest of the graphs will be built starting from graphs in $\mathscr{G}_0$. We also need to have a set of rules which dictate, if a graph G_1 in $\mathscr{G}$ is given, how to produce a new graph G_2 in $\mathscr{G}$. Since we are only interested in rules that are similar to our earlier results, we impose an extra condition on these rules that G_1 and G_2 should not differ too much.

To be a little more precise, for a fixed number ε, let us say that G_1 and G_2 are *ε-close* if each G_i has a set X_i of at most ε vertices and such that $G_1 - X_1$ is isomorphic to $G_2 - X_2$. Let us say that $\mathscr{G}$ can be *ε-generated* from $\mathscr{G}_0$ if, for every graph G in $\mathscr{G}$, there exists a sequence $G_0, G_1, \ldots, G_t$ of graphs in $\mathscr{G}$ such that $G_0 \in \mathscr{G}_0$, $G_t = G$, and any two consecutive terms in the sequence are ε-close. From our discussion in Section 1 we can say that: the class of r-regular graphs, for even r, can be $(r+1)$-generated from the unique one-vertex r-regular graph; and, the class of r-regular graphs, for odd r, can be $2r$-generated from the class of two-vertex r-regular graphs. A general problem is to characterize all classes $\mathscr{G}$ that can be ε-generated, for some ε, from a finite class $\mathscr{G}_0$. Here, we only study regular graphs.

For each pair of nonnegative integers r and g, let $\mathscr{G}_{r,g}$ be the class of r-regular graphs of girth at least g, where the girth of a forest is considered as ∞. A classical result of Erdos and Sachs [7] says that $\mathscr{G}_{r,g}$ is not empty. Let us fix a graph $G_{r,g}$ in $\mathscr{G}_{r,g}$ with the least number of vertices.

Proposition. *$\mathscr{G}_{r,g}$ can be ε-generated from $\{G_{r,g}\}$, where ε depends only on r and g.*

Proof. The result is trivial when $r \leqslant 1$, and thus we assume that $r \geqslant 2$. By our early results in this paper we may also assume that $g \geqslant 3$. Let L be obtained from $G_{r,g}$ by deleting an edge, say ab. The two vertices a and b are called the *roots* of L.

Let G be a graph and let F be a set of edges of G. We construct a new graph $L(G,F)$ as follows. First, for each edge e in F, we take a copy L_e of L such that all the copies and G are mutually vertex disjoint. Then, for each edge $e = xy \in F$, we delete e and add two new edges xa_e and yb_e, where a_e and b_e are the roots of L_e. Notice that the resulting graph $L(G,F)$ has $|V(G)| + |F| \cdot |V(G_{r,g})|$ vertices.

Claim 1. *If G is r-regular, then so is $L(G,F)$.*

From the above construction it is clear that, for each vertex in $V(G)$, its degree in G is the same as its degree in $L(G,F)$. Thus the claim follows.

Claim 2. *If the girth of $G \setminus F$ is at least g, then the girth of $L(G,F)$ is at least g.*

We need to show that every cycle C of $L(G,F)$ has length at least g. This is clear if C is completely contained in $G \setminus F$ or in some L_e. If C is not contained in $G \setminus F$ and not in any L_e, then C contains an edge of the form xa_e or yb_e, for some $e = xy \in F$. Notice that xa_e and yb_e form an edge-cut of $L(G,F)$, thus the cycle C must contain both of these two edges. It follows that part of C is a path P in L_e, between its two roots. Since adding the edge $a_e b_e$ to L_e results a graph of girth at least g, we conclude that P must have length at least $g-1$. Therefore, C has length greater than g and the claim is proved.

Let $O = S$ when r is even and $O = \mathrm{DS}^+$ when r is odd. Let F_1 be a set of edges of an r-regular G_1, let G_2 be obtained from G_1 by applying operation O once, and let F_2 be the union of $E(G_2) \cap F_1$ and $E(G_2) - E(G_1)$. For $i = 1, 2$, let $H_i = L(G_i, F_i)$. Let $\varepsilon = 2r|V(G_{r,g})|$.

Claim 3. *H_1 and H_2 are ε-close.*

Let $Z = V(G_1) - V(G_2)$, $E^- = E(G_1) - E(G_2)$, and $E^+ = E(G_2) - E(G_1)$. Let X_1 be the union of Z and $V(L_e)$, for all $e \in F_1 \cap E^-$. Let X_2 be the union of $V(L_e)$, for all $e \in E^+$. Since $G_1 \setminus E^- - Z = G_2 \setminus E^+$, it follows that $H_1 - X_1 = H_2 - X_2$. Notice that $|Z| \leqslant 2$, $|E^+| \leqslant r-1$, and $|E^-| \leqslant 2r-1$. It follows that $|X_i| \leqslant \varepsilon$, for $i = 1, 2$, and thus the claim is proved.

Claim 4. *If $G = (V, E)$ is an r-regular graph on at most two vertices, then $L(G,E)$ has at most ε vertices.*

Clearly, G has at most r edges. Thus $L(G,E)$ has at most $r|V(G_{r,g})| + 2 \leqslant \varepsilon$ vertices.

Now, let $G \in \mathscr{G}_{r,g}$. From our discussion in Section 1 we know that there is a sequence $G_0, G_1, \ldots, G_t$ of r-regular graphs such that G_0 is G, G_t has at most two vertices, and each G_i, where $1 \leqslant i \leqslant t$, is obtained from G_{i-1} by applying operation O once. Notice that, in each G_i, there are two kinds of edges: those that are edges of $G = G_0$ and those that are created when we split vertices. Let F_i be the set of the second kind of edges in G_i and let $H_i = L(G_i, F_i)$. Let $H_{t+1} = G_{r,g}$. Clearly, $H_0 = G_0 = G$, as $F_0 = \emptyset$. Furthermore, by Claim 1 and Claim 2, every graph H_i belongs $\mathscr{G}_{r,g}$. Finally, by Claim 3 and Claim 4, any two consecutive terms in the sequence $H_0, H_1, \ldots, H_t, H_{t+1}$ are ε-close. Thus the proposition is proved. □

The value of ε given in this proof is certainly not the best possible and, in fact, it might be very far away from the real value. The importance of this proposition is that it tells us that there does exist a procedure, the kind of procedure we had in mind, that generate all graphs in $\mathscr{G}_{r,g}$. The remaining problem is to find a better one.

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