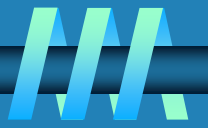
Greedy algorithms

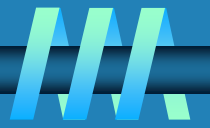


Optimization problems solved through a sequence of choices that are:

- *feasible*
- *locally optimal*
- *irrevocable*

Not all optimization problems can be approached in this manner!

Applications of the Greedy Strategy



- Optimal solutions:
  - change making
  - Minimum Spanning Tree (MST)
  - Single-source shortest paths
  - simple scheduling problems
  - Huffman codes

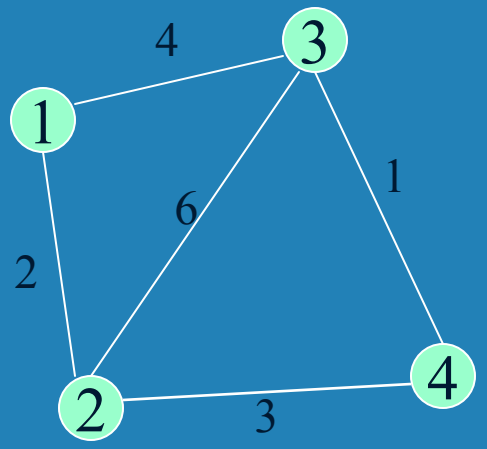
- Approximations:
  - Traveling Salesman Problem (TSP)
  - Knapsack problem
  - other combinatorial optimization problems

Minimum Spanning Tree (MST)

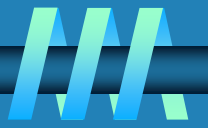
- ***Spanning tree*** of a connected graph *G*: a connected acyclic subgraph of *G* that includes all of *G*'s vertices.

- ***Minimum Spanning Tree*** of a weighted, connected graph *G*: a spanning tree of *G* of minimum total weight.

- **Example:**

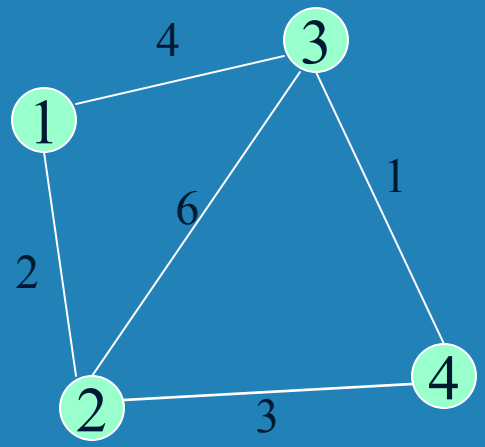


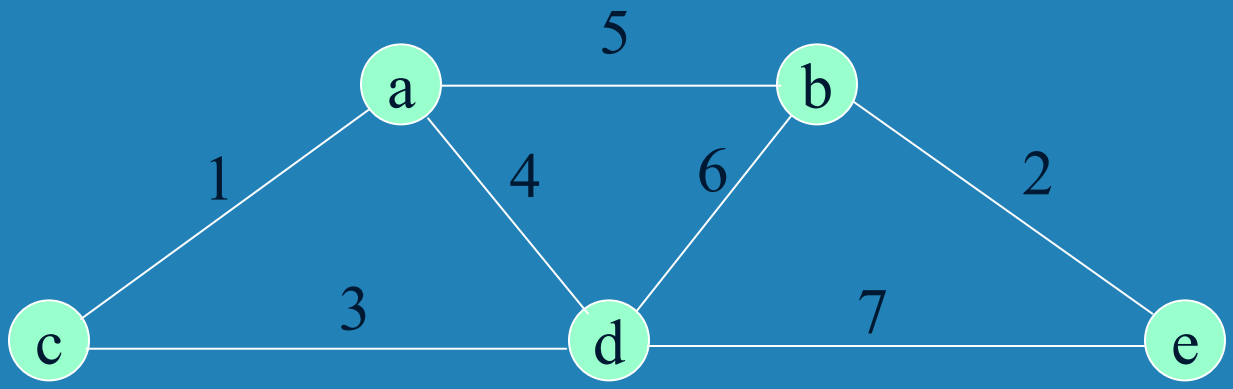
Prim's MST algorithm



- Start with tree consisting of one vertex
- "grow" tree one vertex/edge at a time to produce MST
  - Construct a series of expanding subtrees T_1, T_2, …
- at each stage construct T_{i+1} from T_i: add minimum weight edge connecting a vertex in tree (T_i) to one not yet in tree
  - choose from "fringe" edges
  - (this is the "greedy" step!)
- algorithm stops when all vertices are included

Examples:





Notes about Prim's algorithm

- Need to prove that this construction actually yields MST
- Need priority queue for locating lowest cost fringe edge: use *min-heap*
- *Efficiency:* For graph with n vertices and m edges:

$$(n - 1 + m) \log n$$

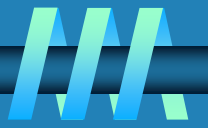
number of stages (min-heap deletions)

number of edges considered (min-heap insertions)

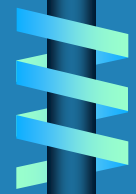
insertion/deletion from min-heap

$$\Theta(m \log n)$$

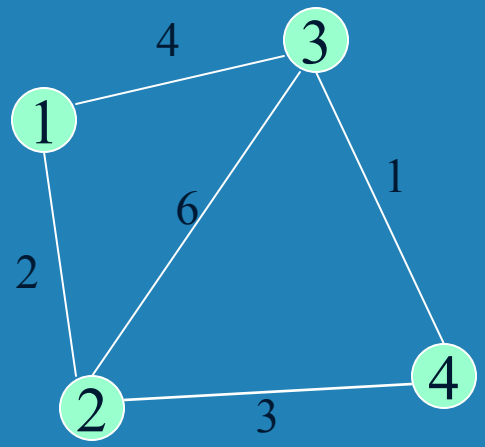
Another Greedy algorithm for MST: Kruskal

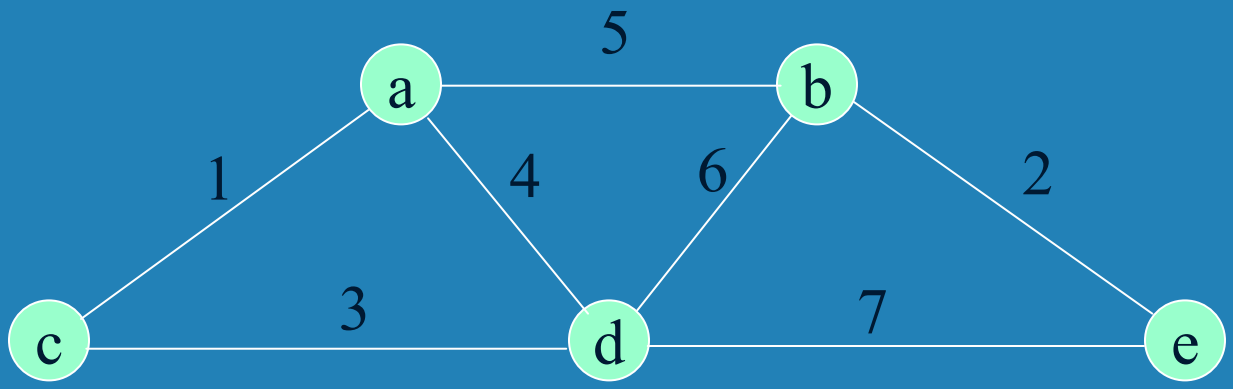


- Start with empty forest of trees
- "grow" MST one edge at a time
  - intermediate stages usually have forest of trees (not connected)
- at each stage add minimum weight edge among those not yet used that does not create a cycle
  - edges are initially sorted by increasing weight
  - at each stage the edge may:
    - expand an existing tree
    - combine two existing trees into a single tree
    - create a new tree
  - need efficient way of detecting/avoiding cycles
- algorithm stops when all vertices are included

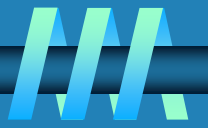


Examples:





Notes about Kruskal's algorithm



- Algorithm looks easier than Prim's but is
  - harder to implement (checking for cycles!)
  - less efficient $\Theta(m \log m)$
- *Cycle checking:* a cycle exists iff edge connects vertices in the same component.
- *Union-find* algorithms – see section 9.2

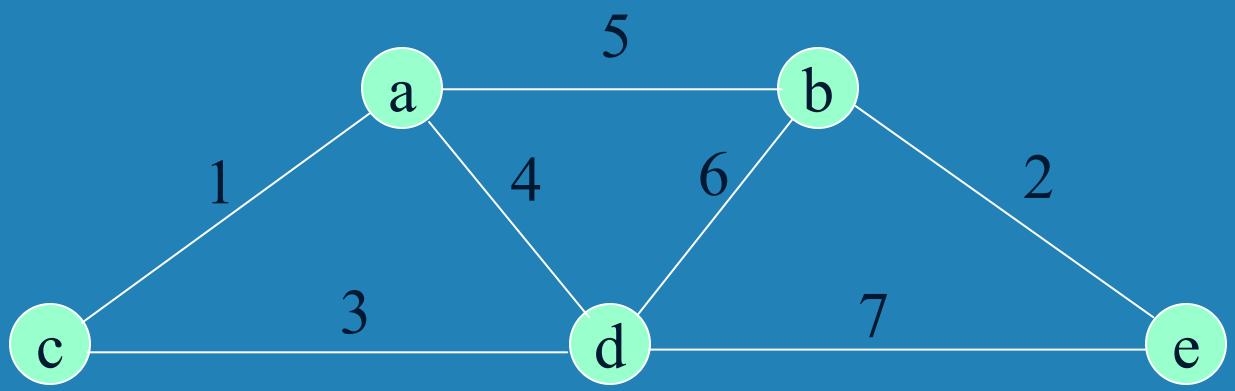
Shortest paths-Dijkstra's algorithm

- ***Single Source Shotest Paths Problem:*** Given a weighted graph G, find the shortest paths from a source vertex s to each of the other vertices.

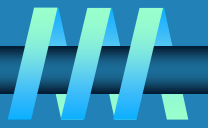
- Dijkstra's algorithm: Similar to Prim's MST algorithm, with the following difference:
  - Start with tree consisting of one vertex
  - "grow" tree one vertex/edge at a time to produce MST
    - Construct a series of expanding subtrees T_1, T_2, …
  - Keep track of shortest path from source to each of the vertices in T_i
  - at each stage construct T_{i+1} from T_i: add ~~minimum weight edge~~ connecting a vertex in tree (T_i) to one not yet in tree
    - choose from "fringe" edges
    - (this is the "greedy" step!)

    edge (*v*,*w*) with lowest *d*(*s*,*v*) + *d*(*v*,*w*)
  - algorithm stops when all vertices are included

Example:



Notes on Dijkstra's algorithm



- Doesn't work with negative weights
- Applicable to both undirected and directed graphs
- Efficiency: