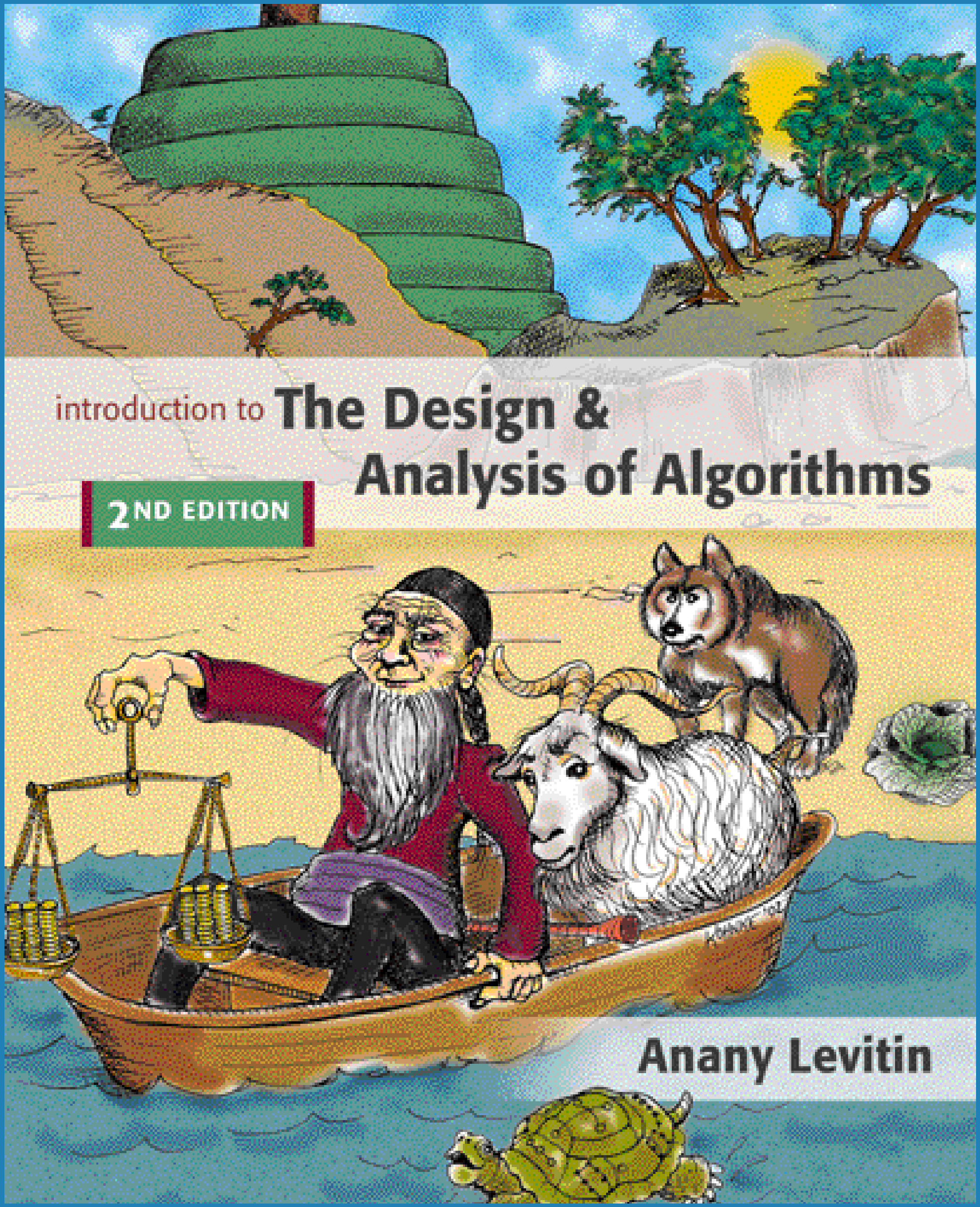
Chapter 8

Dynamic Programming



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Dynamic Programming

Dynamic Programming is a general algorithm design technique for solving problems defined by recurrences with overlapping subproblems

- Invented by American mathematician Richard Bellman in the 1950s to solve optimization problems and later assimilated by CS

- "Programming" here means "planning"

- Main idea:
  - set up a recurrence relating a solution to a larger instance to solutions of some smaller instances
  - solve smaller instances once
  - record solutions in a table
  - extract solution to the initial instance from that table

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Example: Fibonacci numbers

- **Recall definition of Fibonacci numbers:**

$$F(n) = F(n-1) + F(n-2)$$
$$F(0) = 0$$
$$F(1) = 1$$

- **Computing the n^{th} Fibonacci number recursively (top-down):**

$$F(n)$$

$$F(n-1) \quad + \quad F(n-2)$$

$$F(n-2) \quad + \quad F(n-3) \qquad F(n-3) \quad + \quad F(n-4)$$

...

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Example: Fibonacci numbers (cont.)

Computing the n^{th} Fibonacci number using bottom-up iteration and recording results:

$F(0) = 0$
$F(1) = 1$
$F(2) = 1+0 = 1$
…
$F(n\text{-}2) =$
$F(n\text{-}1) =$
$F(n) = F(n\text{-}1) + F(n\text{-}2)$

0	1	1	.	. .	$F(n\text{-}2)$	$F(n\text{-}1)$	$F(n)$

Efficiency:
- time
- space

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Examples of DP algorithms

- Computing a binomial coefficient
- Warshall's algorithm for transitive closure
- Floyd's algorithm for all-pairs shortest paths
- Constructing an optimal binary search tree
- Some instances of difficult discrete optimization problems:
  - traveling salesman
  - knapsack

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Computing a binomial coefficient by DP

Binomial coefficients are coefficients of the binomial formula:

$$(a + b)^n = C(n,0)a^n b^0 + \ldots + C(n,k)a^{n-k}b^k + \ldots + C(n,n)a^0 b^n$$

Recurrence: $C(n,k) = C(n-1,k) + C(n-1,k-1)$ for $n > k > 0$

$C(n,0) = 1, \quad C(n,n) = 1$ for $n \geq 0$

Value of $C(n,k)$ **can be computed by filling a table:**

	0	1	2	. . .	k-1	k
0	1					
1	1	1				
.						
.						
.						
n-1					$C(n-1,k-1)$	$C(n-1,k)$
n						$C(n,k)$

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Computing C(n,k): pseudocode and analysis

```
ALGORITHM Binomial(n, k)
    //Computes C(n, k) by the dynamic programming algorithm
    //Input: A pair of nonnegative integers n ≥ k ≥ 0
    //Output: The value of C(n, k)
    for i ← 0 to n do
        for j ← 0 to min(i, k) do
            if j = 0 or j = i
                C[i, j] ← 1
            else C[i, j] ← C[i − 1, j − 1] + C[i − 1, j]
    return C[n, k]
```

Time efficiency: Θ(*nk*)

Space efficiency: Θ(*nk*)

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Knapsack Problem by DP

Given n items of

integer weights: $w_1 \quad w_2 \; \ldots \; w_n$

values: $v_1 \quad v_2 \; \ldots \; v_n$

a knapsack of integer capacity W

find most valuable subset of the items that fit into the knapsack

Consider instance defined by first i items and capacity j ($j \leq W$).

Let $V[i,j]$ be optimal value of such instance. Then

$$V[i,j] = \begin{cases} \max\{V[i-1,j],\; v_i + V[i-1,j-w_i]\} & \text{if } j-w_i \geq 0 \\ V[i-1,j] & \text{if } j-w_i < 0 \end{cases}$$

Initial conditions: $V[0,j] = 0$ and $V[i,0] = 0$

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Knapsack Problem by DP (example)

Example: Knapsack of capacity $W = 5$

item	weight	value
1	2	\$12
2	1	\$10
3	3	\$20
4	2	\$15

capacity j

		0	1	2	3	4	5
	0						
$w_1 = 2, v_1 = 12$	1						
$w_2 = 1, v_2 = 10$	2						
$w_3 = 3, v_3 = 20$	3						
$w_4 = 2, v_4 = 15$	4						?

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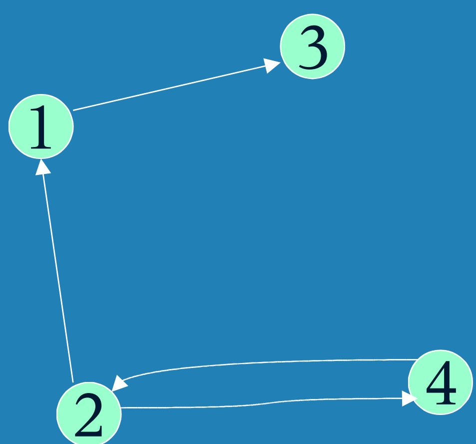
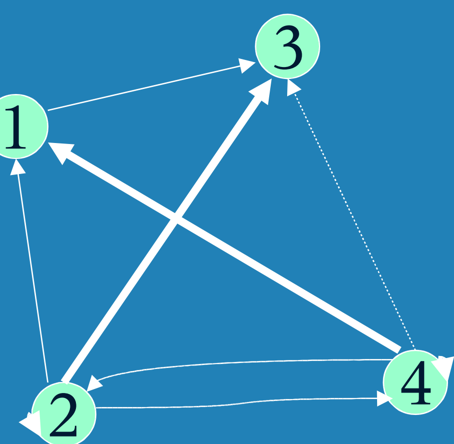
Warshall's Algorithm: Transitive Closure

- **Computes the transitive closure of a relation**
- **Alternatively: existence of all nontrivial paths in a digraph**
- **Example of transitive closure:**



0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0



0	0	1	0
1	**1**	**1**	1
0	0	0	0
1	1	**1**	**1**

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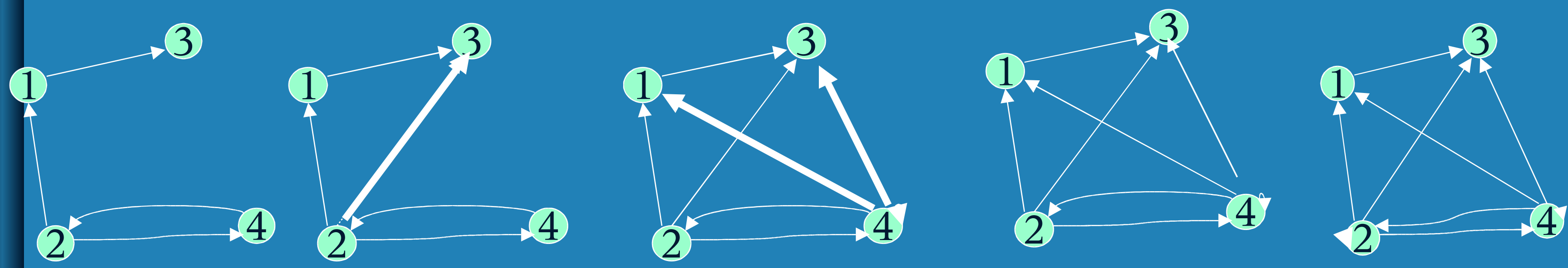
Warshall's Algorithm

Constructs transitive closure T as the last matrix in the sequence of n-by-n matrices $R^{(0)}, \ldots, R^{(k)}, \ldots, R^{(n)}$ where

$R^{(k)}[i,j] = 1$ iff there is nontrivial path from i to j with only first k vertices allowed as intermediate

Note that $R^{(0)} = A$ (adjacency matrix), $R^{(n)} = T$ (transitive closure)



$R^{(0)}$

0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0

$R^{(1)}$

0	0	1	0
1	0	**1**	1
0	0	0	0
0	1	0	0

$R^{(2)}$

0	0	1	0
1	0	1	1
0	0	0	0
1	1	**1**	**1**

$R^{(3)}$

0	0	1	0
1	0	1	1
0	0	0	0
1	1	1	1

$R^{(4)}$

0	0	1	0
1	**1**	1	1
0	0	0	0
1	1	1	1

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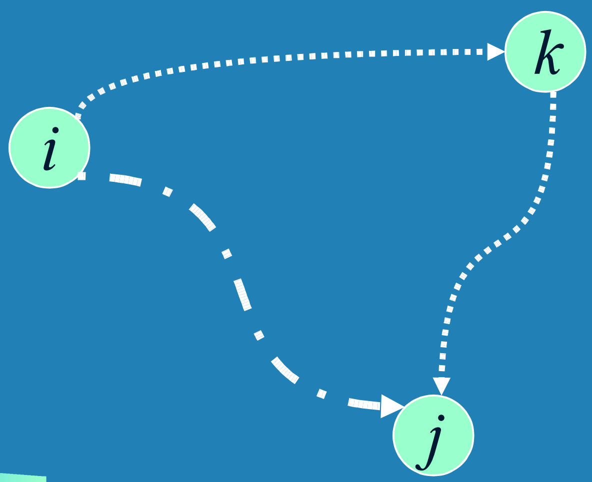
Warshall's Algorithm (recurrence)

On the k-th iteration, the algorithm determines for every pair of vertices i, j if a path exists from i and j with just vertices 1,…,k allowed as intermediate

$$R^{(k)}[i,j] = \begin{cases} R^{(k-1)}[i,j] & \text{(path using just } 1 ,\ldots,k\text{-1)} \\ \quad \text{or} & \\ R^{(k-1)}[i,k] \text{ and } R^{(k-1)}[k,j] & \text{(path from } i \text{ to } k \\ & \text{and from } k \text{ to } i \\ & \text{using just } 1 ,\ldots,k\text{-1)} \end{cases}$$



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Warshall's Algorithm (matrix generation)

Recurrence relating elements $R^{(k)}$ to elements of $R^{(k-1)}$ is:

$$R^{(k)}[i,j] = R^{(k-1)}[i,j] \text{ or } (R^{(k-1)}[i,k] \text{ and } R^{(k-1)}[k,j])$$

It implies the following rules for generating $R^{(k)}$ from $R^{(k-1)}$:

Rule 1 If an element in row i and column j is 1 in $R^{(k-1)}$, it remains 1 in $R^{(k)}$

Rule 2 If an element in row i and column j is 0 in $R^{(k-1)}$, it has to be changed to 1 in $R^{(k)}$ if and only if the element in its row i and column k and the element in its column j and row k are both 1's in $R^{(k-1)}$

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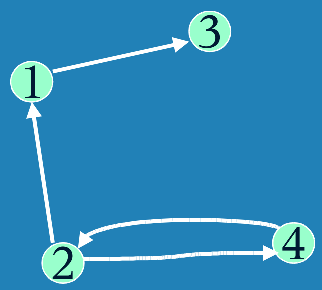
Warshall's Algorithm (example)



$$R^{(0)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

$$R^{(1)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

$$R^{(2)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} \end{matrix}$$

$$R^{(3)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

$$R^{(4)} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & \mathbf{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

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Warshall's Algorithm (pseudocode and analysis)

ALGORITHM *Warshall*$(A[1..n, 1..n])$

//Implements Warshall's algorithm for computing the transitive closure
//Input: The adjacency matrix A of a digraph with n vertices
//Output: The transitive closure of the digraph

$R^{(0)} \leftarrow A$
for $k \leftarrow 1$ **to** n **do**
  for $i \leftarrow 1$ **to** n **do**
    for $j \leftarrow 1$ **to** n **do**
      $R^{(k)}[i, j] \leftarrow R^{(k-1)}[i, j]$ **or** $(R^{(k-1)}[i, k]$ **and** $R^{(k-1)}[k, j])$
return $R^{(n)}$

Time efficiency: $\Theta(n^3)$

Space efficiency: Matrices can be written over their predecessors

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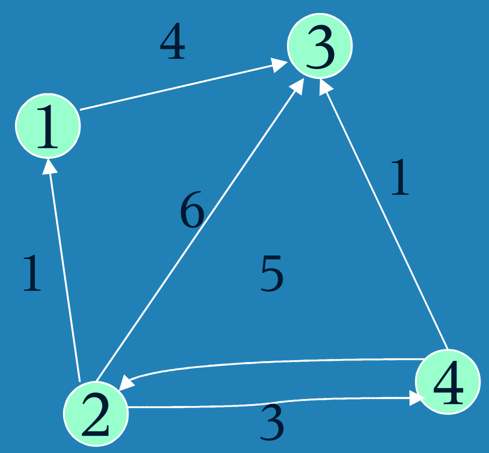
Floyd's Algorithm: All pairs shortest paths

Problem: In a weighted (di)graph, find shortest paths between every pair of vertices

Same idea: construct solution through series of matrices $D^{(0)}, \ldots, D^{(n)}$ using increasing subsets of the vertices allowed as intermediate

Example:



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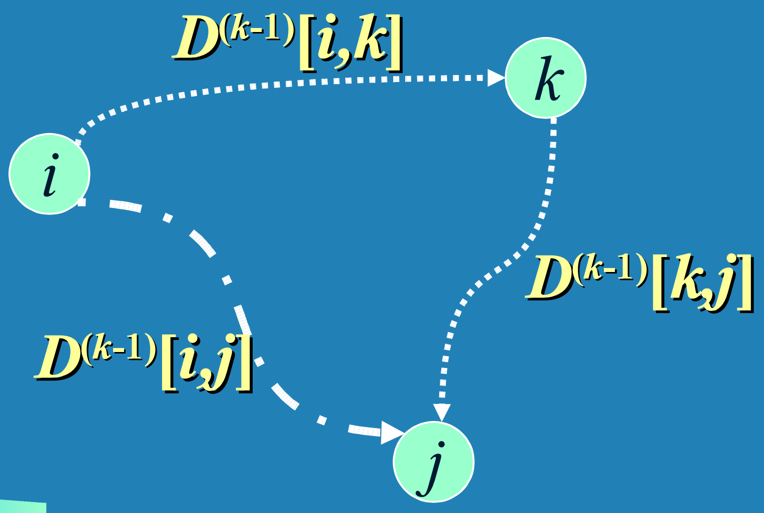
Floyd's Algorithm (matrix generation)

On the *k*-th iteration, the algorithm determines shortest paths between every pair of vertices *i, j* that use only vertices among 1,…,*k* as intermediate

$$D^{(k)}[i,j] = \min \{D^{(k-1)}[i,j],\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j]\}$$



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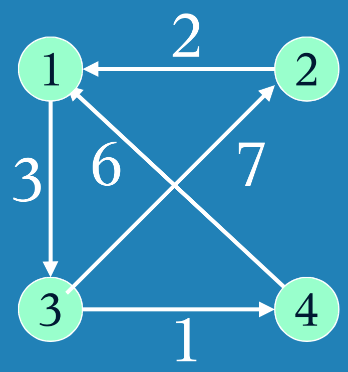
Floyd's Algorithm (example)



$$D^{(0)} = \begin{matrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{matrix}$$

$$D^{(1)} = \begin{matrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \mathbf{5} & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \mathbf{9} & 0 \end{matrix}$$

$$D^{(2)} = \begin{matrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \mathbf{9} & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{matrix}$$

$$D^{(3)} = \begin{matrix} 0 & \mathbf{10} & 3 & \mathbf{4} \\ 2 & 0 & 5 & \mathbf{6} \\ 9 & 7 & 0 & 1 \\ 6 & \mathbf{16} & 9 & 0 \end{matrix}$$

$$D^{(4)} = \begin{matrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ \mathbf{7} & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{matrix}$$

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Floyd's Algorithm (pseudocode and analysis)

```
ALGORITHM  Floyd(W[1..n, 1..n])
    //Implements Floyd's algorithm for the all-pairs shortest-paths problem
    //Input: The weight matrix W of a graph with no negative-length cycle
    //Output: The distance matrix of the shortest paths' lengths
    D ← W //is not necessary if W can be overwritten
    for k ← 1 to n do
        for i ← 1 to n do
            for j ← 1 to n do
                D[i, j] ← min{D[i, j], D[i, k] + D[k, j]}
    return D
```

Time efficiency: $\Theta(n^3)$

Space efficiency: Matrices can be written over their predecessors

Note: Shortest paths themselves can be found, too

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Optimal Binary Search Trees

Problem: Given n keys $a_1 < \ldots < a_n$ and probabilities $p_1 \le \ldots \le p_n$ searching for them, find a BST with a minimum average number of comparisons in successful search.

Since total number of BSTs with n nodes is given by C(2n,n)/(n+1), which grows exponentially, brute force is hopeless.

Example: What is an optimal BST for keys A, B, C, and D with search probabilities 0.1, 0.2, 0.4, and 0.3, respectively?

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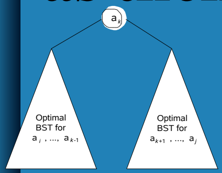
DP for Optimal BST Problem

Let $C[i,j]$ be minimum average number of comparisons made in T[i,j], optimal BST for keys $a_i < \ldots < a_j$, where $1 \le i \le j \le n$. Consider optimal BST among all BSTs with some a_k ($i \le k \le j$) as their root; T[i,j] is the best among them.



$$C[i,j] =$$

$$\min_{i \le k \le j} \{p_k \cdot 1 +$$

$$\sum_{s=i}^{k-1} p_s \text{ (level } a_s \text{ in T}[i,k\text{-}1] + 1) +$$

$$\sum_{s=k+1}^{j} p_s \text{ (level } a_s \text{ in T}[k+1,j] + 1)\}$$

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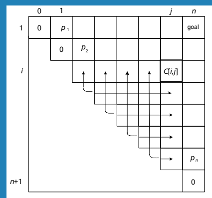
DP for Optimal BST Problem (cont.)

After simplifications, we obtain the recurrence for $C[i,j]$:

$$C[i,j] = \min_{i \le k \le j} \{C[i,k-1] + C[k+1,j]\} + \sum_{s=i}^{j} p_s \quad \text{for } 1 \le i \le j \le n$$

$$C[i,i] = p_i \quad \text{for } 1 \le i \le j \le n$$



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Example:	key	*A*	*B*	*C*	*D*
	probability	0.1	0.2	0.4	0.3

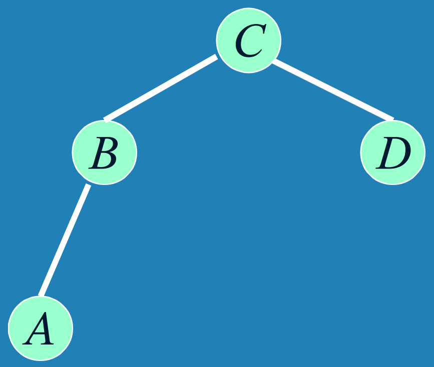
The tables below are filled diagonal by diagonal: the left one is filled using the recurrence

$$C[i,j] = \min_{i \le k \le j} \{C[i,k\text{-}1] + C[k+1,j]\} + \sum_{s=i}^{j} p_s, \quad C[i,i] = p_i;$$

the right one, for trees' roots, records *k*'s values giving the minima

i \ j	0	1	2	3	4
1	0	.1	.4	1.1	1.7
2		0	.2	.8	1.4
3			0	.4	1.0
4				0	.3
5					0

i \ j	0	1	2	3	4
1		1	2	3	3
2			2	3	3
3				3	3
4					4
5					



optimal BST

Optimal Binary Search Trees

ALGORITHM *OptimalBST*($P[1..n]$)

```
//Finds an optimal binary search tree by dynamic programming
//Input: An array P[1..n] of search probabilities for a sorted list of n keys
//Output: Average number of comparisons in successful searches in the
//           optimal BST and table R of subtrees' roots in the optimal BST
for i ← 1 to n do
    C[i, i − 1] ← 0
    C[i, i] ← P[i]
    R[i, i] ← i
C[n + 1, n] ← 0
for d ← 1 to n − 1 do //diagonal count
    for i ← 1 to n − d do
        j ← i + d
        minval ← ∞
        for k ← i to j do
            if C[i, k − 1] + C[k + 1, j] < minval
                minval ← C[i, k − 1] + C[k + 1, j]; kmin ← k
        R[i, j] ← kmin
        sum ← P[i]; for s ← i + 1 to j do sum ← sum + P[s]
        C[i, j] ← minval + sum
return C[1, n], R
```

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Analysis DP for Optimal BST Problem

Time efficiency: $\Theta(n^3)$ but can be reduced to $\Theta(n^2)$ by taking advantage of monotonicity of entries in the root table, i.e., $R[i,j]$ is always in the range between $R[i,j\text{-}1]$ and $R[i+1,j]$

Space efficiency: $\Theta(n^2)$

Method can be expended to include unsuccessful searches

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